HAMILTON-JACOBI-BELLMAN EQUATION FOR STOCHASTIC OPTIMAL CONTROL: APPLICATIONS TO SPACECRAFT ATTITUDE CONTROL

BY

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THESIS

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ABSTRACT

This study aims to address the problem of attitude control of spacecraft in presence of thrust uncertainty, which leads to stochastic accelerations. Spacecraft equipped with electric propulsion and other low thrust mechanisms, often experience random fluctuations in thrust. These stochastic processes arise from sources such as uncertain power supply output, varying propellant flow rate, faulty thrusters, etc. Mission requirements and mass/fuel limitations demand an optimal and proactive method of control to mitigate the thrust uncertainty and parasitic torque. Stabilizing stochastic optimal control of the satellite attitude dynamics is derived through formulation of the Hamilton-Jacobi-Bellman equation associated with a stochastic differential equation. The solution to the Hamilton-Jacobi-Bellman partial differential equation is approximated through the method of Al'brekht [1]. Extension of Al'brekht method for a stochastic system was first presented in [2]; detailed derivations of linear and nonlinear stochastic control laws along with their analytical and numerical analyses are presented in this thesis. A planning method is then discussed to lower the error due to local nature of the control.

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CHAPTER 1
Introduction

1.1 *Background and Problem Overview*

Understanding thrust-induced disturbance is critical to the design of attitude controllers in need of precision pointing, as well as, reduction of fuel consumption and actuator wear. In this study, system disturbance is modeled proportional to the generated thrust. We propose an optimal control strategy that reduces the thrust uncertainty effects by directly accounting for the uncertainty in the dynamics. Considering the generated uncertainty by each thruster enables us to embed the uncertainty information directly in the proposed control law. In this manner, we formulate an optimal controller that adjusts its behavior based on the best-known information on the thrust-induced disturbance and the given optimality criteria.

This study is motivated by the growing applications of continuous thrust technologies such as low thrust electric propulsion (EP). Due to advances in EP technologies, recent missions have started to consider EP as a viable option for attitude control. In contrast to rather traditional momentum exchange devices, EP thrusters are not massive, nor suffer from wheel friction instabilities and needs of desaturation of accumulated momentum [3]. One example is LISA Pathfinder's attitude control system which solely relies on varying continuous thrust through use of Field Emission Electric Propulsion (FEEP) thrusters [4].

Furthermore, use of smaller satellites and CubeSats has become favorable recently. Smaller satellites are cheaper to manufacture and are capable of carrying out valuable science missions. In fact, use of electric and non-electric thrusters as actuators, has been shown viable for smaller spacecraft attitude control systems. The 6U CubeSats used in the Mars Cube One (MarCO) mission, for instance, use thrusters to power their attitude control system. These CubeSats have been used as a communication relay to Earth. Among other future CubeSat class science missions, Lunar IceCube is proposed to include Electric Propulsion for actuation.

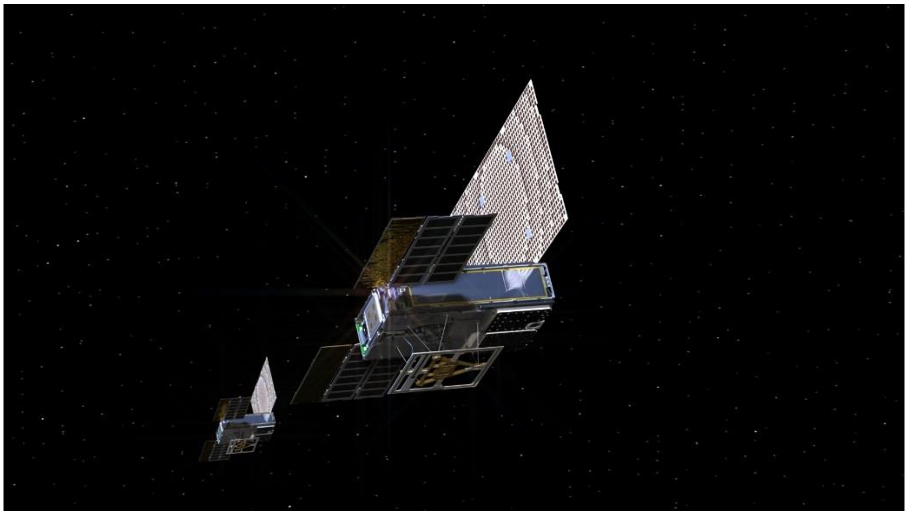

Figure 1. Mars Cube One Mission CubeSats (Image Courtesy of NASA/JPL-Caltech [5])

One of the challenges in low thrust EP is thrust fluctuations. Low-thrust propulsion engines usually operate for a long range of time continuously, and thrust can fluctuate over time, as shown by experimental studies [6],[7],[8], and [9]. Particularly, Nicolini et al. [7] demonstrated the effects of increasing thrust level on the increasing error accuracy in a thrust measurement experiment for the FEEP thrusters. Similarly, this relationship has also been shown in Abbot [10] in the study of low thrust propulsion techniques in satellite attitude control. The variations in thrust are directly proportional to the discharge current fluctuations which have been found to be 8% - 13% of the nominal value [11].

In order to achieve an efficient attitude control performance under such thrust fluctuations, an optimal control law that takes the thrust fluctuations into consideration is needed. As low thrust propulsion engines operate for a long range of time continuously, any fluctuations in thrust can be modeled as stochastic processes. Several studies have previously addressed actuator uncertainty. In an influential work, McLane [12] derived the solution of the linear regulator problem for thrust-dependent noise in a physical system. Similarly, in the study of stochastic Hill's equations, Ostoja-Starzewski and Longuski [13] modeled the thrust as an additive random process. Gustafson [14] provided the numerical methods for the optimal feedback control of linear spacecraft system with thrusters. Zhao et al. [15] investigated the attitude stabilization of a stochastic spacecraft system under additive disturbance. The problem of actuator uncertainty and error has also been addressed in [16], [17],[18], [19],[20], [21], and [22].

In response to challenges arising in attitude control of uncertain systems, this work aims to solve a nonlinear quadratic regulator formulation for the attitude dynamics assuming a continuous varying thrust model. The stabilization of rotational rates of the spacecraft under uncertainty is considered. An optimal control law is proposed which achieves a desired minimum fuel consumption criterion, while reducing thrust uncertainty effects. This is specifically useful in applications such as proximity operations, in-orbit servicing, and precision instrument pointing in science missions where state error is highly undesirable. Moreover, the minimum fuel criterion is essential for satellites with smaller fuel and energy supplies. The attitude dynamics of the spacecraft under input uncertainty is modeled as a stochastic differential equation (SDE). A *Hamilton-Jacobi-Bellman* (HJB) equation associated with a SDE is then formulated ensuring stability and optimality if a solution exists. Linear and nonlinear control laws are sought, although, due to difficulties arising in solving the HJB directly, a powerful power series-based method is used: The Al'brekht method [1] provides a local solution to the HJB equation associated with a nonlinear differential equation. Al'brekht's method expands the dynamics, running cost, value function, and the control as power series which are later substituted into the HJB equation. The expanded HJB is then organized in different orders of the state variable. The quadratic order forms the Riccati equation. Eventually at cubic order and every order higher than that, a homologicaltype equation is solved. In this manner solutions to different orders of value function and the optimal control are sought. Although the mentioned method yields a closed form solution to the HJB, the solution holds locally, and the optimality error increases further from the origin. Thus, solutions may be considered close to optimal locally.

In this work, the Al'brekht method is extended for an infinite horizon HJB equation corresponding to a stochastic optimal control problem in which noise enters the system through the control input. Linear and nonlinear stochastic optimal controls are solved, and algebraic solutions are presented. Simulating the angular velocity stabilization of a 6U CubeSat, the acquired stochastic and deterministic controls are analyzed and compared. The contributions of this research are summarized as: I) Developing a control method to account for the actuator uncertainty effects in attitude stabilization applications; and II) Extending the Al'brekht method for the HJB equation corresponding to the stochastic optimal control problem.

1.2 *Organization of Thesis*

In **chapter 1**, the motivation behind this research is highlighted. General, descriptions of the physical problem, and a brief overview of the state-space method, and attitude control system are provided for readers with different backgrounds. In **chapter 2**, the governing equations of motion: the Euler rigid body rotational dynamics, are derived. The modeling of the physical problem is then presented and divided into two complementary sections: modeling of a deterministic system, and modeling of a stochastic system with multiplicative noise. The general concept of controllability of linear and nonlinear system is summarized. The provided controllability conditions are a useful tool in analyzing the attitude control systems, especially when discussing the actuator count. **Chapter 3** gives the derivations of optimal control and the dynamic programing principle. In the same chapter, the Itô's Lemma and the diffusion generator are derived. The derivations of Hamilton-Jacobi-Bellman equation associated with a deterministic dynamical system, as well as a stochastic dynamical system, are then reviewed. In **chapter 4**, the method of Al'brekht [1] which is the core of this thesis, is presented. The Al'brekht method provides a solution to the Hamilton-Jacobi-Bellman equation locally and is a powerful tool when dealing with nonlinear systems. **Chapter 5** contains the main contribution of this thesis: the Al'brekht method is extended for a stochastic system with multiplicative control noise. Optimal control for the uncertain model of chapter 2 is computed. Solvability conditions of the stochastic control are provided. **Chapter 6** provides the numerical results of the control derived in chapter 5. Several conclusions have been drawn based on the numerical and analytical result of the proposed method. Benefits of using a stochastic optimal controller have also been outlined. In conclusion, a trajectory planning method has been discussed as future work, which may reduce the optimality error for the stochastic Al'brekht method.

1.3 *Intro to Modern Control Systems*

Fueled by the Cold War, 1950s saw the rise of modern control cultivated in the aerospace industry. Although the idea of feedback in engineering is more than a century old, the field of modern control and the state-space approach was first spearheaded through the works of Rudolf Kálmán mid-twentieth century. Furthermore, the classical works of Lyapunov and Poincaré in stability theory and dynamical systems have served as an enabling power in modern control theory. The idea of modern control and state-space representation is to describe systems and their processes as differential equations. This allows the evolution of systems to be described by all of their internal variables, inputs, and outputs. In a physical system specifically, it is often desired to drive a system's parameter to a desired value. As an example, consider the first order time-invariant linear ordinary differential equation

$$m\dot{\omega} = -c\omega + \tau$$

where, m is the mass, ω is angular velocity, and variable τ is the generated torque by a motor, or servo. In the differential equation above, $-c\omega$ describes a drag force, while $m\dot{\omega}$ is rate of change of angular momentum. To rewrite the system in a state-space form, let $x = \omega$, and $u = \tau$, where the variable x is called the *state* and u the *input* (or control) variable.

$$\dot{x}_t = \left(-\frac{c}{m}\right) x_t + (\frac{1}{m}) u(t)$$

Subscript t implies that x evolves with time. For the input u, this is communicated by writing $u(t)$. Renaming $A = -\frac{c}{m}$, and $B = \frac{1}{m}$, then the one-dimensional system is written as

$$\dot{x}_t = Ax_t + Bu(t) \tag{1.1}$$

Note that the linear system (1.1) is 1-dimensional because it is described by only one state variable. We further say that the control system (1.1) is time-invariant if A and B do not vary with time. Next, let $u(t) = 0 \ \forall \ t \geq 0$ such that (1.1) becomes $\dot{x}_t = Ax_t$, which is the *uncontrolled* dynamics equation. The trajectory solution of this equation is the exponential decay $x_t = e^{At}x(0)$, where $x(0)$ is the *initial condition* of the differential equation (the value of $\omega(t)$ at $t = 0$). It can be justified that the system is stable and decays to zero because of damping effect due to the friction force $-\frac{c}{m}x_t$. The decay will happen if no control torque is inputted. On the other hand, a linear feedback control will have the form $u(t) = kx_t$, where k is called the *control gain* and is appropriately calculated to achieve stability and other desired criteria. The control may be chosen in a way to increase ω to a desired value, or to *stabilize* the system to zero angular velocity before the uncontrolled decay due to friction.

Often it is desired to know the evolution of θ along with its rate $\dot{\theta}$. It is straightforward to relate the angular velocity to the rotational angle θ by setting $\omega = \dot{\theta}$. Through defining x_t as a *vector* $x_t = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, where $x \in \mathbb{R}^2$, the state-space system can be written in matrix form containing θ

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

Renaming the matrices, $A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix}$, and $B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$, then the two-dimensional system becomes

$$\dot{x} = Ax_t + Bu(t) \tag{1.2}$$

In contrast to equation (1.1), x in equation (1.2) is a vector in $\mathbb{R}^2$ and is referred to as the *state vector*. The dimension of the *input vector* is an important quantity which governs the controllability properties of the system; this is discussed with more details in section 2.4 of this chapter. Note that for a general system $\dot{x} = Ax_t + Bu(t)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$, constants n, and m are determined by design, and the assumed model of the problem.

1.4 *Attitude Control System*

Attitude control is a subdiscipline of the guidance, navigation, and control (GNC) engineering. Specifically, attitude refers to the orientation of the spacecraft with respect to a reference frame, and attitude control is controlling the rotational rates and the orientation of the spacecraft. In space engineering, attitude control system is formally referred to as attitude control and determination system (ADCS). This naming refers to two components of a spacecraft which are either tasked with "determining" the current rates and orientation, or "controlling" the angular rates and steering the system to a new orientation. This is accomplished by groups of sensors and actuators. Sensors such as star trackers, sun sensors, etc. are used to determine the orientation with respect to an external point or object. Other onboard devices such as inertial measurement units (IMUs) and gyroscopes are used for measuring the system states such as angular rates, and forces, to determine the attitude. Two main strategies used for controlling the attitude are the active and passive control methods. An Example of passive strategy is use of gravity gradient for attitude control. Operation of devices such as thrusters and momentum wheels is considered to be an active control strategy.

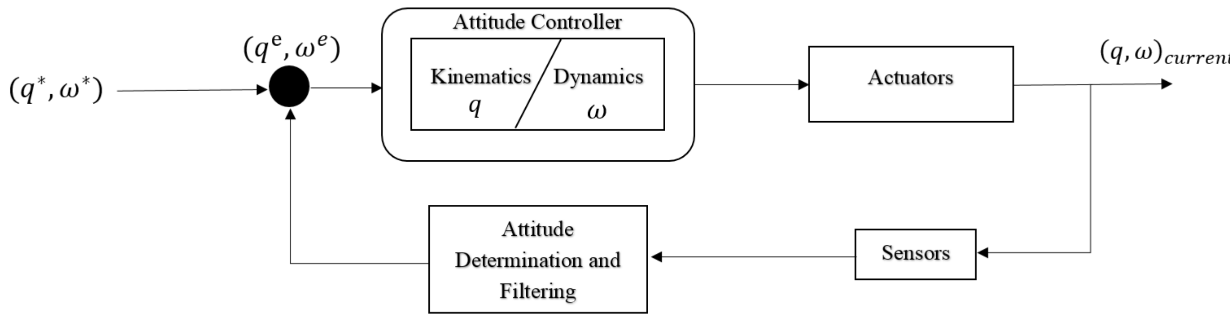

Figure 2. A Generic Attitude Control Loop

In general, an attitude control task can be thought of two control problems when one separates the kinematics and the dynamics of a spacecraft. The kinematics is concerned with characterizing the pointing of the spacecraft and driving the system to a specific orientation, i.e. controlling the angles with respect to a reference frame. There are several different parametrization of attitude kinematics some of which are: the Euler angles, quaternions, classical Rodrigues parameters, refined Rodrigues parameters, etc. Euler angles have been known to be "more intuitive to work with", though with the disadvantage of a phenomenon known as the gimbal lock. Quaternions however are known to avoid the gimbal lock. The evolution of quaternions is described by the following bilinear differential equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \omega_1 q_4 - \omega_2 q_3 + \omega_3 q_2 \\ \omega_1 q_3 + \omega_2 q_4 - \omega_3 q_1 \\ -\omega_1 q_2 + \omega_2 q_1 + \omega_3 q_4 \end{bmatrix}$$

$$\dot{q}_4 = -\frac{1}{2}(\omega_1 q_1 + \omega_2 q_2 + \omega_3 q_3)$$

where ω_i, $i = 1,2,3$ are the angular rates around the three body axes, and q_i, $i = 1,2,3,4$ are the quaternion coordinates parametrizing the attitude kinematics. The goal of kinematics control law is to achieve a desired (q_1^*, q_2^*, q_3^*). Notice that the input to these equations is the ω vector, which is governed by the Euler rotational dynamics equations (see section 2.1). This brings us to the second control task which is controlling the dynamics. By this, we mean driving the spacecraft to the desired angular rates $(\omega_1^*, \omega_2^*, \omega_3^*)$. The desired angular rates are either determined by the kinematics system, or, in specific modes such as detumbling, where desired angular velocity of zero is to be achieved (stabilization). In general, the two dynamics and kinematics systems are coupled; for instance, a general linear control law will have the form: $u = -k_1\omega - k_2 q_{1:3}$, for both systems. In this thesis, we will only study the control of the dynamics system and leave the coupled control of kinematics and dynamics as a suggestion for future research. Because the actuators directly affect the dynamics equations and noise is first propagated through the dynamics, we will study the actuator uncertainty in context of dynamics equations. Further details on kinematics parametrization and control can be found in references [23], and [24]. For a control problem dealing with both kinematics and the dynamics systems see for example ref. [25].

Attitude control systems can also be divided into categories of spin stabilized and three-axis control. In this study, we will specifically provide control algorithms for the three-axis control system and a family of actuators known as reaction control system (RCS). Reaction control systems use jets and thrusters to actively control the attitude by ejecting a form of mass to create a force (thrust). When this force is not pointed towards the center of mass, a torque is produced (i.e. when the lever arm is nonzero). While in general, there are three main families of actuators: RCS, the momentum devices, and magnetic devices, advantages of thrusters compared to other families of actuators is their great response time, higher maneuvering speed [24], and their relatively good accuracy. However, the disadvantage of RCS is in fuel limitation, error and uncertainty due to aging and cycling of the components. Common thruster types used in attitude control applications are the hot gas (hydrazine) and cold gas thrusters which can produce a thrust on the range of 0.5 to 9000 Newtons [26]. Electric propulsion engines and electric microthrusters however can produce a smaller and more precise thrust which makes them an ideal choice for use in smaller spacecraft such as CubeSats. Moreover, modern electric propulsion systems are capable of producing a relatively continuous thrust profile. A currently in development thruster engine, known as the Variable Specific Impulse Magnetoplasma Rocket (VASIMR), has the capability of producing variable thrust. Although a larger engine, the properties that VASIMR offer are useful for space missions in need of fuel efficiency and precision.

When it comes to attitude control and rotational motion using RCS, the thrusters are usually operated in pairs. This is because a thruster is only capable of producing a one-sided force vector. Moreover, a single force vector around the center of causes a rotation, as well as some translational motion which might be unwanted. To counter this translational motion and to produce a purely rotational motion around a single axis, a pair of thrusters, pointing in opposite directions are used -- see for instance figure 4. This in turn complicates the calculation of force required for attitude maneuvers. More importantly, this means that an algorithm is needed to convert the torque commands calculated by a control system to thruster activation time [24]. For an example of this algorithm, see sidi [24] section 9.2.2.

Traditionally, the number of required thrusters to fully control the attitude of a spacecraft in all axes is six or more thrusters, i.e. three pairs, however, more modern control systems, based on some controllability assumptions, have been able to lower the number of the thrusters needed to steer the spacecraft to different orientations -- for example see Sidi [24] section 9.5 or section 2.5

of this thesis. Clearly, the location and placement of a thruster also determines the amount of generated torque by a single thruster. The longer the lever arm of a thruster is measured from the center of mass, the more torque can be generated by the thruster. However, if the lever arm is decreased, then thrusters would be able to produce a more precise attitude change by producing a larger force, at the cost of losing more fuel [24].

In general, algorithms and control laws presented in this thesis, and most of other works dealing with attitude control system, assume that actuators are able to produce a continuous variable thrust. Aside from the class of electric propulsion systems which are capable of this task to some degrees, most propulsion systems do not operate continuously. In fact, traditional non-electric thrusters are operated in an on/off manner. This means that when thrusters are turned on, they are capable of producing one level of force only. Hence, to compensate for this operational limitation, the conversion of a continuously variable commanded control torque to a series of constant magnitude pulses is needed [27]. This is accomplished using a technique known as the pulse width modulation (PWM). In practice, the operation of a thruster is divided into sample periods t_k to t_{k+1}. Then the computed control torque T_c is kept constant during that sample period. The following equation is used to compute the amount of time for which a thruster is turned on

$$t_{p,k} = \frac{\tau_c(t_k)(t_{k+1} - t_k)}{\tau_t}$$

where τ_t is the generated torque by the thruster, and $\tau_c(t_k)$ is the commanded control during the sample period starting at t_k. Then, the average applied torque on $[t_k, t_{k+1}]$ is equal to the average control torque calculated by the controller [27]. A more detailed treatment of this practice is presented in references [27], [24], and [23].

CHAPTER 2
Governing Equations

2.1 *Euler Rotational Rigid Body Dynamics Equations*

In this section, the Euler rigid body dynamics equations are derived. These nonlinear equations are specifically important to this study as they govern the rotational dynamics of the satellite. Let us start by defining few preliminaries. We shall first begin by deriving an expression for the angular momentum of a rigid body in $\mathbb{R}^3$. Then the moment of inertia tensor and its diagonalizability conditions will be shown, and finally the Euler rigid body dynamics equations will be derived. Euler rotational equations are necessary for modeling the attitude dynamics.

Consider the origin of a reference frame, point O, which is attached to a rigid body consisting of i particles of mass m_i. The angular momentum of the system of i particles with respect to point O is then given by the summation

$$H_O = \sum_i r_i \times m_i \dot{r}_i \tag{2.1}$$

where r_i is the position vector from point O to the particle of mass m_i, and $\dot{r}_i$ is its rate of change. By Rotation Axis Theorem [28], the velocity of vector r_i can be expressed as

$$\dot{r}_i = (\dot{r}_i)_{rel} + \omega \times r_i \tag{2.2}$$

where ω is the absolute angular velocity of the rigid body, and $(\dot{r}_i)_{rel}$ is the velocity of r_i measured in reference frame O, fixed to the body. Moreover, since the system of particles is a rigid body, the $(\dot{r}_i)_{rel}$ is zero, thus, equation (2.2) becomes

$$\dot{r}_i = \omega \times r_i \tag{2.3}$$

It must be pointed out that $\dot{r}_i$ is the velocity of the reference (body) frame with origin O, measured in a non-rotating (fixed) frame. Combining equation (2.3) and the angular momentum equation (2.1), the angular momentum with respect to point O becomes

$$H_O = \sum_i m_i (r_i \times (\omega \times r_i)) \tag{2.4}$$

Next, let the mass m_i be written as ρdV, where ρ is the (uniform) density of the rigid body, and dV is an increment of the volume of mass m_i. The summation (2.4) is then rewritten as an integral over the volume of the rigid body

$$H_O = \int \rho (r \times (\omega \times r))\, dV \tag{2.5}$$

Let $\beta = \{e_1, e_2, e_3\}$ be the basis of the vector space $\mathbb{R}^3$. Rewriting vectors ω and r in basis β, $[\omega]_\beta = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$, and $[r]_\beta = r_1 e_1 + r_2 e_2 + r_3 e_3$, we may evaluate $r \times (\omega \times r)$ as

$$\begin{aligned} r \times (\omega \times r) = \quad & \{(r_2^2 + r_3^2)\omega_1 + (-r_1 r_2)\omega_2 + (-r_1 r_3)\omega_3\} e_1 \\ & + \{(-r_2 r_1)\omega_1 + (r_1^2 + r_3^2)\omega_2 + (-r_2 r_3)\omega_3\} e_2 \\ & + \{(-r_3 r_1)\omega_1 + (-\mathrm{r}_3 \mathrm{r}_2)\omega_2 + (r_1^2 + r_2^2)\omega_3\} e_3 \end{aligned} \tag{2.6}$$

The moment of inertia tensor in standard basis becomes

$$[L_I]_\beta^\beta = I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

where, $L_I : \mathbb{R}^3 \to \mathbb{R}^3$ and entries of I are given in the table 7 of Appendix A. Combining equations (2.5), and (2.6), and recognizing the moment of inertia terms, angular momentum of the rigid body becomes

$$H = \sum_i \sum_j I_{ij} \omega_j e_i \tag{2.7}$$

$$= \{I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3\}e_1 + \{I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3\}e_2$$

$$+ \{I_{31}\omega_1 + \mathrm{I}_{32}\omega_2 + I_{33}\omega_3\}e_3$$

More concisely in vector form, (2.7) is expressed as

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{2.8}$$

We have now derived the angular momentum about the origin of a reference frame, point O [28]. The goal is to arrive at equations governing the rotational dynamics of a rigid body. Hence, we may use the obtained equation to write the angular momentum about the center of mass of a rigid body by choosing O as the center of the gravity, CG. It is well known that rate of change of angular momentum $\dot{H}$, is equal to the applied external moment M

$$\dot{H} = M$$

To evaluate $\dot{H}$, we apply the Rotation Axis Theorem [28] to vector H. The resulting $\dot{H}$ is the rate of change of the angular momentum vector in the absolute frame

$$\dot{H} = \left(\dot{H}\right)_{rel} + (\omega \times H) \tag{2.9}$$

where, $\left(\dot{H}\right)_{rel} = \dot{H}_1 + \dot{H}_2 + \dot{H}_3$ is the rate of change of H measured in the body frame about point $O = CG$. Differentiating the entries of vector H in (2.8), we obtain

$$\begin{aligned} \dot{H}_1 &= I_{11}\dot{\omega}_1 + I_{12}\dot{\omega}_2 + I_{13}\dot{\omega}_3 \\ \dot{H}_2 &= I_{21}\dot{\omega}_1 + I_{22}\dot{\omega}_2 + I_{23}\dot{\omega}_3 \\ \dot{H}_3 &= I_{31}\dot{\omega}_1 + I_{32}\dot{\omega}_2 + I_{33}\dot{\omega}_3 \end{aligned} \tag{2.10}$$

Similarly, evaluating the cross product $\omega \times H$, we have

$$\omega \times H = (H_3\omega_2 - H_2\omega_3)e_1 + (H_1\omega_3 - H_3\omega_1)e_2 + (H_2\omega_1 - H_1\omega_2)e_3 \tag{2.11}$$

Hence substituting (2.10) and (2.11) in equation (2.9) and realizing that $M = M_1e_1 + M_2e_2 + M_3e_3 = \left(\dot{H}\right)_{rel} + (\omega \times H)$, we obtain the following:

$$M_1 = I_{11}\dot{\omega}_1 + I_{12}\dot{\omega}_2 + I_{13}\dot{\omega}_3 + (H_3\omega_2 - H_2\omega_3)$$

$$M_2 = I_{21}\dot{\omega}_1 + I_{22}\dot{\omega}_2 + I_{23}\dot{\omega}_3 + (H_1\omega_3 - H_3\omega_1)$$

$$M_3 = I_{31}\dot{\omega}_1 + I_{32}\dot{\omega}_2 + I_{33}\dot{\omega}_3 + (H_2\omega_1 - H_1\omega_2)$$

Let us now substitute in the expressions H_1, H_2, and H_3 from equation (2.8) so that the equations for M_1, M_2, and M_3 become

$$\begin{aligned} M_1 &= I_{11}\dot{\omega}_1 + I_{12}\dot{\omega}_2 + I_{13}\dot{\omega}_3 + I_{31}\omega_1\omega_2 + \mathrm{I}_{32}\omega_2\omega_2 + I_{33}\omega_3\omega_2 - I_{21}\omega_1\omega_3 \\ &\quad - I_{22}\omega_2\omega_3 - I_{23}\omega_3\omega_3 \\ M_2 &= I_{21}\dot{\omega}_1 + I_{22}\dot{\omega}_2 + I_{23}\dot{\omega}_3 + I_{11}\omega_1\omega_3 + I_{12}\omega_2\omega_3 + I_{13}\omega_3\omega_3 - I_{31}\omega_1\omega_1 \\ &\quad - \mathrm{I}_{32}\omega_2\omega_1 - I_{33}\omega_3\omega_1 \\ M_3 &= I_{31}\dot{\omega}_1 + I_{32}\dot{\omega}_2 + I_{33}\dot{\omega}_3 + I_{21}\omega_1\omega_1 + I_{22}\omega_2\omega_1 + I_{23}\omega_3\omega_1 - I_{11}\omega_1\omega_2 \\ &\quad - I_{12}\omega_2\omega_2 - I_{13}\omega_3\omega_2 \end{aligned} \tag{2.12}$$

Above equations contain the general moment of inertia tensor, I, in the body frame. However, it is often desired to calculate these equations in *principal axes*, where I is a diagonal matrix. Such matrix is obtainable from a *diagonalizable* general moment of inertia matrix.

Consider the full rank moment of inertia matrix $[I]_\beta^\beta$, where β is the standard basis in $\mathbb{R}^3$. Matrix $[L_I]_\beta^\beta$ is diagonalizable if there exists another basis $\alpha = \{v_1, v_2, v_3\}$ of R^3, such that

$$[L_I]_\alpha^\alpha = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{ for } \lambda_i,\ i = 1,2,3$$

and $L_I(v_i) = \lambda_i v_i$, $i = 123$. Clearly, we have that the diagonal entries of matrix $[L_I]_\alpha^\alpha$ are the eigenvalues of matrix I. We may then express $[L_I]_\alpha^\alpha$ as

$$[L_I]_\alpha^\alpha = [I_{3\times3}]_\beta^\alpha [L_I]_\beta^\beta [I_{3\times3}]_\alpha^\beta = Q^{-1} I Q = I_P$$

where, $[I_{3\times3}]_\alpha^\beta = Q$ is the matrix representative of the identity map $I_{3\times3}$ with respect to bases α and β, and $[I_{3\times3}]_\beta^\alpha = Q^{-1}$ is its inverse, and I_P is the principal moment of inertia matrix. It must be noted that columns of $[I_{3\times3}]_\alpha^\beta$'s are the eigenvectors of $[L_I]_\beta^\beta$.

Theorem 1. [29] Consider the matrix $I \in \mathbb{R}^{3\times3}$. We say that I is diagonalizable if

1) $\det(I - \lambda I_{3\times3})$ splits over $\mathbb{R}$
2) For each eigenvalue λ, the geometric multiplicity is equal to the algebraic multiplicity.

The above conditions are summarized as follows: The first condition requires $\det(I - \lambda I_{3\times3})$ to split over $\mathbb{R}$. This implies that $\det(I - \lambda I_{3\times3})$ must factor completely. For instance, the expression $(a - \lambda_1)(b - \lambda_2)(c - \lambda_3)$ splits over $\mathbb{R}$, for $a, b, c \in \mathbb{R}$. In the second condition, the *Algebraic multiplicity* refers to the number of times, λ appears as a root of the characteristic polynomial of I. The *Geometric multiplicity* however refers to the dimension of the eigenspace of I, E_λ. If the two are equal, and additionally $\det(I - \lambda I_{3\times3})$ factors completely, then I is diagonalizable.

Assuming that I is diagonalizable with the principal moments of inertia I_{11}, I_{22}, and I_{33} as the diagonal entries

$$[L_I]_\alpha^\alpha = I_P = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

then equations (2.12) are simplified to

$$M_1 = I_{11}\dot{\omega}_1 + I_{33}\omega_3\omega_2 - I_{22}\omega_2\omega_3$$

$$M_2 = I_{22}\dot{\omega}_2 + I_{11}\omega_1\omega_3 - I_{33}\omega_3\omega_1 \tag{2.13}$$

$$M_3 = I_{33}\dot{\omega}_3 + I_{22}\omega_2\omega_1 - I_{11}\omega_1\omega_2$$

where, $I_{12} = I_{21} = I_{31} = I_{13} = I_{23} = I_{32} = 0$. The resulting equations are known as the Euler equations of motion, or Euler dynamics equations. They describe the time rate of change of ω in terms of the applied moment M, principal moment of inertia matrix I_P, and the angular velocity vector ω. Rearranging the terms of (2.13), and factoring the common angular rate terms, a more useful and familiar form of (2.13) is

$$\dot{\omega}_1 = \frac{(I_{22} - I_{33})}{I_{11}}\omega_2\omega_3 + \frac{M_1}{I_{11}}$$

$$\dot{\omega}_2 = \frac{(I_{33} - I_{11})}{I_{22}}\omega_3\omega_1 + \frac{M_2}{I_{22}} \tag{2.14}$$

$$\dot{\omega}_3 = \frac{(I_{11} - I_{22})}{I_{33}}\omega_1\omega_2 + \frac{M_3}{I_{33}}$$

Few observations can be drawn from this form of the Euler equations. First consideration is that the three equations are nonlinear. In fact, the angular rates of the two opposite axes, affect the third axis. For instance, if $\omega_2 = \omega_3 = 0$, then $\omega_3 = M_3$. It is also important to note that in absence of external moments, two equal principal moments of inertia would result in a constant rate of the corresponding axis. For instance, suppose, $I_{11} = I_{22}$, then ω_3 will have a constant value. In such cases of symmetry, angular rate of the constant axis is changed through an applied moment M. Ultimately, we consider the moments M_1, M_2, and M_3 as the “inputs” to these equations. In the next section, we will use the physical meaning of the moment M to define the control input u, and further develop a model for the spacecraft attitude dynamics.

2.2 *Spacecraft Attitude Control System Modeling*

In this section, the deterministic state-space representation of the spacecraft attitude dynamics will be discussed. Consider the Euler equations of motion (2.14). For simplicity, we will adapt few briefer notations and assumptions. From now on, we'll be using the letter x, $x \in \mathbb{R}^3$, as the state vector instead of ω for angular rate, and I instead of I_P for the principal moment of inertia matrix.

We now present the calculation of torque due to a single thruster. Assume that $r = r_1 e_1 + r_2 e_2 + r_3 e_3$ is the vector from the center of gravity (center of the body frame) to the thruster of interest. To describe the orientation of the thrusters in the body frame, we employ a spherical coordinate frame as shown below.

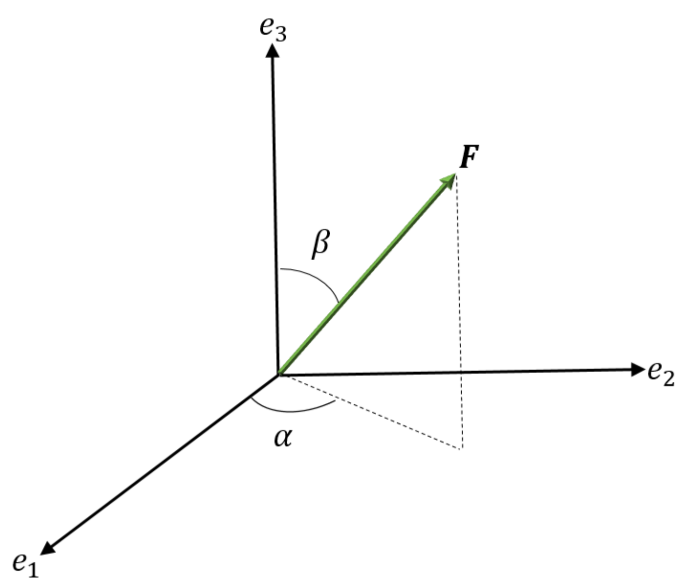

Figure 3. Thruster Force Vector in Spherical Coordinates

Constant angles α and β are the thruster azimuth and elevation angles [24]. Specifically, angle α is measured form e_1 axis to the projection of vector F onto the $e_1 \times e_2$ plane. Angle β is measured from the e_3 axis to the vector F as shown in figure 3. Thus, the generated torque from a single thruster is calculated as

$$\tau = r \times F = b\acute{\boldsymbol{F}} = \begin{bmatrix} r_2\cos(\beta) - r_3\sin(\alpha)\sin(\beta) \\ r_3\cos(\alpha)\sin(\beta) - r_1\cos(\beta) \\ r_1\sin(\alpha)\sin(\beta) - r_2\cos(\alpha)\sin(\beta) \end{bmatrix}\acute{\boldsymbol{F}} \tag{2.15}$$

where, $\acute{\boldsymbol{F}}$ is the scalar magnitude of the force generated by the thruster, and the force vector F is

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) \\ \cos(\beta) \end{bmatrix}\acute{\boldsymbol{F}} \tag{2.16}$$

As stated in the previous section, thrusters are typically operated in pairs in attitude control maneuvers [24]. For further simplification, we assume that each thruster pair is mounted symmetrically, that is, the vectors from the center of mass of the spacecraft to each thruster are of equal length. Suppose that the spacecraft is equipped with i pair of thrusters. Hence, consider the moments M_i, $i = 1, 2, 3$ to be the moments of force, or *torque* τ generated by the i^{th} thruster pair. The forces due to thruster 1 and 2 of the i^{th} pair are denoted by F_{i_1} and F_{i_2} respectively.

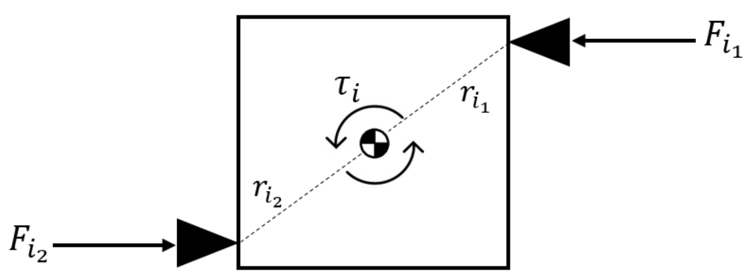

Figure 4. Produced Torque by a Thruster Pair

For instance, for the lever arms $r_i = r_{i_1} = -r_{i_2}$, the generated torque by the i^{th} thruster pair is calculated as $\tau_i = r_{i_1} \times F_{i_1} + r_{i_2} \times F_{i_2} = r_{i_1} \times \left(\|F_{i_1}\| + \|F_{i_2}\|\right)\frac{F_{i_1}}{\|F_{i_1}\|}$. Let us denote expression $\left(\|F_{i_1}\| + \|F_{i_2}\|\right)\frac{F_{i_1}}{\|F_{i_1}\|}$ by F_i, that is the net vector force generated by the i^{th} thruster pair. Then, $\tau_i = r_i \times F_i$ is the torque generated by the i^{th} pair, and the total generated torque τ is summation of torques generated by all the thruster pairs. For instance, for m thruster pairs, the torque vector is given by

$$\tau = \sum_{i=1}^{m} \tau_i = \sum_{i=1}^{m} b_i \acute{\boldsymbol{F}}_i \tag{2.17}$$

where $\acute{\boldsymbol{F}}_i = \|F_{i_1}\| + \|F_{i_2}\|$ is the scalar magnitude of the force generated by the i^{th} thruster pair, and b_i is given by (2.15). Expressing the torque vector by the state-space notation, τ becomes

$$\tau = \sum_{i=1}^{m} b_i \acute{\boldsymbol{F}}_i = \sum_{i=1}^{m} b_i u(t)_i = bu(t) \tag{2.18}$$

where, $u \in \mathbb{R}^m$ is the control vector, and $b : \mathbb{R}^n \to \mathbb{R}^m$, is a real valued n by m matrix. The columns of b, namely b_i, give the orientation of each thruster pair in terms of angles α and β. In fact, b_i vectors are the axes about which the corresponding control torques $\|b_i\|u_i$ are applied [30]. We consider the vectors b_i to be time invariant by assumption. The entries of vector u, describe the generated net force by each thruster pair. Substituting bu_t as the generated moment in equation (2.14), the Euler equations of motion become

$$\dot{x}_t = f(x_t) + I^{-1}\tau = f(x_t) + I^{-1}bu(t)$$

$$f(x) = \begin{bmatrix} \frac{I_{22} - I_{33}}{I_{11}} x_2 x_3 \\ \frac{I_{33} - I_{11}}{I_{22}} x_3 x_1 \\ \frac{I_{11} - I_{22}}{I_{33}} x_1 x_2 \end{bmatrix} \tag{2.19}$$

where $f(x_t)$ is the drift vector field. Defining the matrix cross product for $x \in \mathbb{R}^3$, the skew-symmetric matrix $[x]_\times = \left(S(x)\right)^T$ is defined as

$$[x]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{2.20}$$

where $([x]_\times)^T = -[x]_\times$, and $S(x)Ix = \big(x \times (xI)\big) = f(x)$. The spacecraft deterministic attitude dynamics can then be summarized by the following familiar form

$$I\dot{x}_t = S(x_t)Ix_t + \sum_{i=1}^{m} b_i u(t)_i \tag{2.21}$$

$$S(x) = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix} \tag{2.22}$$

For the fully actuated case, $n, m = 3$, in this thesis we assume a spacecraft that is equipped with 3 thrusters. A control system is said to be *underactuated* if $\dim(x) > rank(b)$. For system (2.21) described above, condition $n > m$ will imply that the system is underactuated. In sections 2.4, and 2.5 controllability of different types of control systems will be discussed.

2.3 *Multiplicative Noise and Spacecraft Thrust Uncertainty Modeling*

To model the dynamics with actuator uncertainty, first consider the deterministic system model (2.19), which is given by

$$\dot{x}_t = f(x_t) + Bu(t) = G\big(x_t, u(t)\big) \tag{2.23}$$

where, $B = I^{-1}b$. The main idea is to let generated uncertainty from the i^{th} thruster be modeled as a Gaussian white noise process η_{t_i}, where all the η_{t_i} are independent. The uncertainty due to a thruster pair can be represented as

$$\big(\eta_{t_1} + \eta_{t_2}\big) = \xi_t$$

where ξ_t is a Gaussian mean-zero white noise process. Then we have that

$$u(t)_i = \bar{u}(t)_i(1 + (\xi_i)_t),\, i = 1,\ \dots\ , m$$

and the control vector with multiplicative noise becomes

$$Bu_t = I^{-1} \sum_{i=1}^{m} b_i(\bar{u}_i(t))(1 + (\xi_i)_t) \tag{2.24}$$

where $\bar{u}(t) \in \mathbb{R}^m$ is the nominal control vector. In general, ξ_t accounts for uncertainty in control input, such as thrust magnitude variations.

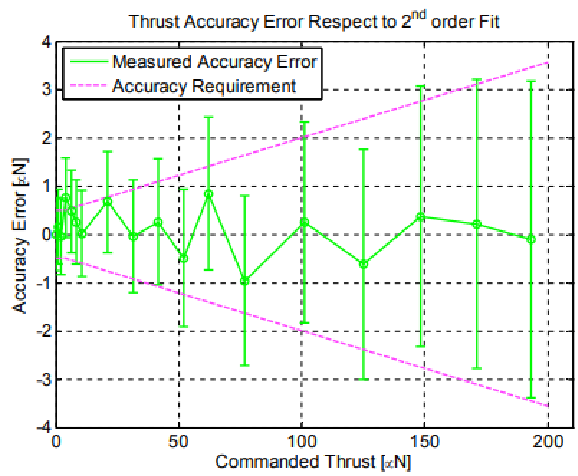

Figure 5. [7] Thrust Accuracy Error Measurements in Study conducted by Nicolini et al. [7]

As opposed to the additive noise model considered in Ref. [13], the multiplicative uncertainty structure provides a more accurate and realistic model where the magnitude of noise generated by the thruster pair is dependent on the magnitude of the control input itself. For instance, a small commanded nominal control $\bar{u}$ will result in $(\xi\bar{u}) \approx 0$ for an arbitrary ξ. Furthermore, it is known that for a measurable function $\sigma(\bar{u}(t))$

$$\int \sigma(\bar{u}(t))\xi_t\, dt \approx \int \sigma(\bar{u}(t))\, dW_t$$

are statistically equivalent [31]. Hence, the differential equation (2.23) is statistically equivalent to

$$x_t = x_o + \int_{t_o}^{t} G(x_s, \bar{u}_s)\, ds + \int_{t_o}^{t} \sigma(\bar{u}_t)\, dW_t \tag{2.25}$$

Given equation (2.25), we may restate (2.23) as a controlled Itô stochastic differential equation (SDE) with a multiplicative noise structure

$$dx_t = G(x_t, \bar{u}(t))\, dt + \sigma(\bar{u}(t)) dW_t \tag{2.26}$$

where $G(x_t, u(t)) = f(x_t) + Bu(t)$ is the vector field containing the dynamics, $W_t\ t \geq 0$ is the m-dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\sigma(u)$ denotes the diffusion coefficient. In the case of spacecraft thrusters with multiplicative noise, the diffusion coefficient is a function of control and is given by

$$\sigma(\bar{u}) = \varepsilon B \begin{bmatrix} \bar{u}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{u}_m \end{bmatrix} \tag{2.27}$$

where, $\varepsilon \geq 0$ is a real parameter scaling the thruster uncertainty effects. The diagonal control matrix of the diffusion coefficient is to make sure that each entry of the m-dimensional Wiener process is associated with its respective $(\bar{u}_i)_t$, $i = 1, \ldots, m$.

2.4 *Controllability and Observability*

In this section, we will review the controllability and observability properties of linear systems. Consider the following general linear control system

$$\begin{aligned} \dot{x}_t &= Ax_t + Bu(t) \\ y_t &= Cx_t \end{aligned} \tag{2.28}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $y \in \mathbb{R}^q$, and $C \in \mathbb{R}^{q \times n}$. Here, the additional equation $y_t = Cx_t$ is known as the *output equation*. In simple terms, the output equation defines the relationship between the current state of the system and its output.

System (2.28) is said to be *controllable* if for every x and every terminal time $T > 0$, there exists a continuous input $u(t)$ $0 \le t \le T$, such that the system is taken from the initial condition at $x(0)$ to $x(T)$ at $t = T$ [32]. To derive the controllability condition, consider the solution to the differential equation (2.28)

$$x_T = \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau = \int_0^T e^{A\tau} Bu(T-\tau) d\tau \tag{2.29}$$

Then, for system (2.28) to be controllable, x_T must span $\mathbb{R}^n$. We have that the matrix exponential $e^{A\tau}$ can be expanded as $e^{A\tau} = I + A\tau + \frac{A^2\tau^2}{2} + \cdots + \frac{A^k\tau^k}{k!} + \cdots$, $k \in \mathbb{N}$. Thus, (2.29) becomes

$$x_T = \int_0^T [I + A\tau + \frac{A^2\tau^2}{2} + \cdots + \frac{A^k\tau^k}{k!} + \cdots] Bu(T-\tau) d\tau \tag{2.30}$$

By Cayley-Hamilton theorem [33], [29], the matrix exponential expansion is then written as

$$I + A\tau + \frac{A^2\tau^2}{2} + \cdots + \frac{A^k\tau^k}{k!} + \cdots = \sum_{i=1}^{n-1} \alpha_i(t) A^i$$

Hence, (2.29) becomes

$$x_T = \int_0^T \left(\sum_{i=1}^{n-1} \alpha_i(\tau) A^i \right) Bu(T-\tau) d\tau$$

$$= \int_0^T [\ B \quad AB \quad A^2B \ \ldots \ A^{n-1}B\ ] \begin{bmatrix} \alpha_o(\tau) \\ \vdots \\ \alpha_{n-1}(\tau) \end{bmatrix} u(T-\tau) d\tau \tag{2.31}$$

$$= [\ B \quad AB \quad A^2B \ \ldots \ A^{n-1}B\ ] \int_0^T \begin{bmatrix} \alpha_o(\tau) \\ \vdots \\ \alpha_{n-1}(\tau) \end{bmatrix} u(T-\tau) d\tau$$

We shall call $\mathcal{C} = [\ B \quad AB \quad A^2B \ \ldots \ A^{n-1}B\ ]$, the *controllability matrix* for linear systems. Hence, x_T can be made equal to any arbitrary specified n-dimensional vector, only if $\mathcal{C}$ has full row rank, i.e. $rank(\mathcal{C}) = \dim(x) = n$ [32].

Linear system (2.28) is *observable* if the initial state $x(0) = x_o$ can be uniquely determined from the knowledge of the input $u(t)$ and the output y_t for $0 \leq t \leq T$. We show the observability condition for the linear system (2.28) by similarly deriving a rank condition for *observability matrix*. Consider the solution of the $\dot{x}_t = Ax$, given by $x_t = e^{At}x_o$. Substituting the trajectory into the equation $y_t = Cx_t$, we obtain the output equation $y_t = Ce^{At}x_o$, as a function of $A \in \mathbb{R}^{n\times n}$. Taking the derivative of the entries of y_t equation $n-1$ times, the following matrix is constructed

$$\begin{bmatrix} y_t = Ce^{At}x_o \\ \dfrac{dy_t}{dt} = ACe^{At}x_o \\ \vdots \\ \dfrac{d^{n-1}y_t}{dt^{n-1}} = A^{n-1}Ce^{At}x_o \end{bmatrix} \tag{2.32}$$

Evaluating the matrix (2.32) at $t = 0$, the matrix is then rewritten as [32]

$$\begin{bmatrix} y_o \\ \dfrac{dy_o}{dt} \\ \vdots \\ \dfrac{d^{n-1}y_o}{dt^{n-1}} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_o \tag{2.33}$$

We identify $\mathcal{O} = [C \;\; CA \;\; \cdots \;\; CA^{n-1}]^T$ as the observability matrix. By definition, it is implied that all the entries of $\left[y_o \;\; \frac{dy_o}{dt} \;\; \frac{d^{n-1}y_o}{dt^{n-1}}\right]^T$ are known when $\mathcal{O}$ is full rank-n. We have that the linear system (2.28) is observable, if and only if the matrix $\mathcal{O}$ is of full column rank [32]. In this thesis, all the presented models and systems are assumed to have satisfied the observability condition. However in practice, it is not a given that observability is satisfied, thus additional work is required. In fact, in this thesis we have made the assumption that all the measurements and state knowledge are perfect and known.

2.5 *Controllability of Spacecraft Gas Jet Thrusters*

In this section, we first consider the general nonlinear system on the smooth n-manifold M

$$\dot{x}_t = f(x_t) + bu(t) \tag{2.34}$$

where $f(x)$ is an arbitrary nonlinear vector field, and $x \in M$. For the special case of spacecraft attitude dynamics, $f(x)$ is given by (2.19), and column vectors b_i are the axes which the torques are applied about (see section 2.2). Then for the attitude dynamics, (2.34) becomes

$$\dot{x}_t = f(x_t) + I^{-1}\sum_{i=1}^{m} b_i u(t)_i \tag{2.35}$$

The controllability conditions for a spacecraft attitude control system equipped with thrusters are derived in Crouch [30] and Isidori [34]. For system (2.35), with external torques $\|b_i\|u_i$, Crouch provided the controllability condition as follows:

Theorem 2. [30] Given a bounded control, and a Poisson stable vector field $f(x)$, system (2.35) is controllable if and only if it is accessible.

Therefore, to analyze the system, we shall first review the definitions of a reachable set, and accessibility. Here, Poisson stability implies that not all trajectories of the system can leave the neighborhood of a Poisson stable point.

A *reachable set* $R(x_o, t)$ [35] of system (2.34), for a given $x_o \in M$ is defined as the set of all $x \in M$ for which there exists an admissible control u, such that there is a trajectory of (2.34) with $x(0) = x_o$, and $x(t) = x$. The reachable set from initial condition x_o at time T is then given by

$$R_T(x_o) = \bigcup_{0 \le t \le T} R(x_o, t) \tag{2.36}$$

Intuitively, this is the set of all the points that are reached by the trajectories of the system given an initial condition $x_o \in M$.

Next, we define the *accessibility algebra* $\mathcal{C}$ [35] as the span of all possible Lie brackets of f and b_i, that is, the smallest Lie algebra on M that contains the vector fields f, and b_i, $i = 1, \dots, m$. The accessibility distribution $\mathcal{C}$ [35] of the general system (2.34) is the distribution generated by the vector fields in $\mathcal{C}$. To give the accessibility distribution of (2.34), we shall first define the notion of *Lie brackets*. The Lie brackets of vector fields f and b [34] is defined as

$$[f, b](x) = \frac{\partial b}{\partial x} f(x) - \frac{\partial f}{\partial x} b(x) \tag{2.37}$$

where $\frac{\partial b}{\partial x}$ and $\frac{\partial f}{\partial x}$ are the Jacobian matrices of b and f respectively, i.e.

$$\frac{\partial b}{\partial x} = \begin{bmatrix} \frac{\partial b_1}{\partial x_1} & \cdots & \frac{\partial b_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_n}{\partial x_1} & \cdots & \frac{\partial b_n}{\partial x_n} \end{bmatrix}, \quad \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

For system (2.34), the accessibility distribution is given by

$$\mathcal{C} = [b_1 \dots \ b_m \ \left[ad_{b_i}^k b_j\right] \ \dots \ \left[ad_f^k b_i\right]] \tag{2.38}$$

where, $ad_f^k b(x) = [f, ad_f^{k-1} b](x)\ k \geq 1$, i.e. $\left[ad_f^k b\right] = \left[f \dots j \dots [f, b]\right]$, and $\left[ad_f^0 b\right] = [b]$. Then system (2.34) is said to be *accessible* from a point x_o if for any $T > 0$, the reachable set $R_T(x_o)$ contains a nonempty open set [35]. This intuitively means that there exists an arbitrary point $x_f \in M$ that is reachable form x_o in finite time T. Moreover, point x_o also needs to be reachable form x_f in finite time T.

Theorem 3. [35] Suppose that f is the smooth vector field of system (2.35). If $\dim[\mathcal{C}(x_o)] = n$, then for any $T > 0$, the set $R_T(x_o)$ contains a nonempty open set.

This implies that the system is accessible from x_o. Moreover, for any $T > 0$, we say that system (2.34) is *small-time locally controllable* from x_o, if x_o is an interior point of $R_T(x_o)$ [36],[35]. We now provide brief computations of accessibility distribution for a spacecraft model (2.35) with external torque actuators, and $x \in \mathbb{R}^3$ [35],[30]:

Case I. Consider a spacecraft for $m = 3$, given by the model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{(I_{22} - I_{33})}{I_{11}} x_2 x_3 \\ \frac{(I_{33} - I_{11})}{I_{22}} x_1 x_3 \\ \frac{(I_{11} - I_{22})}{I_{33}} x_1 x_2 \end{bmatrix} + \begin{bmatrix} \frac{b_1}{I_{11}} \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \frac{b_2}{I_{22}} \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ \frac{b_3}{I_{33}} \end{bmatrix} u_3$$

Case I describe the fully actuated case with application of three control torques (i.e. thruster pairs) u_1, u_2, and u_3. Here, the accessibility distribution $\mathcal{C}$ is given by

$$\mathcal{C}(x) = \begin{bmatrix} \frac{b_1}{I_{11}} & 0 & 0 & 0 & \frac{b_2\, x_3(I_{22} - I_{33})}{I_{11} I_{22}} & \frac{b_3\, x_2\, (I_{22} - I_{33})}{I_{11} I_{33}} \\ 0 & \frac{b_2}{I_{22}} & 0 & -\frac{b_1\, x_3(I_{11} - I_{33})}{I_{11} I_{22}} & 0 & -\frac{b_3\, x_1(I_{11} - I_{33})}{I_{22} I_{33}} \\ 0 & 0 & \frac{b_3}{I_{33}} & \frac{b_1 x_2(I_{11} - I_{22})}{I_{11} I_{33}} & \frac{b_2\, x_1(I_{11} - I_{22})}{I_{22} I_{33}} & 0 \end{bmatrix}$$

Since the three directions b_i are linearly independent, the rank condition is 3, i.e. $\dim \mathcal{C}(x) = 3$. More specifically when the system is at rest,

$$\mathcal{C}(0) = \begin{bmatrix} \frac{b_1}{I_{11}} & 0 & 0 \\ 0 & \frac{b_2}{I_{22}} & 0 \\ 0 & 0 & \frac{b_3}{I_{33}} \end{bmatrix} \tag{2.39}$$

thus, the system is accessible. One can always assume that a fully actuated system is always accessible as long as the control input directions are linearly independent.

Case II. Next consider the case $m = 2$ and the underactuated system $\dot{x} = f(x) + b_1 u_1 + b_2 u_2$. The accessibility distribution $\mathcal{C}(0)$ is computed as

$$\mathcal{C}(0) = \begin{bmatrix} \frac{b_1}{I_{11}} & 0 & 0 \\ 0 & \frac{b_2}{I_{22}} & 0 \\ 0 & 0 & \frac{b_1}{I_{11}} \frac{b_2}{I_{22}} (\frac{I_{11} - I_{22}}{I_{33}}) \end{bmatrix} \tag{2.40}$$

Hence, the system is accessible as long as $I_{11} \neq I_{22}$. It should be noted that $I_{11} = I_{22}$ will result in a constant third axis, i.e. $\dot{x}_3 = 0$. Assuming $I_{11} \neq I_{22}$, we have $\dim \mathcal{C}(0) = 3$, hence the assumed system of case II is accessible.

Case III. Let us now consider the case $m = 1$, and the resulting equation $\dot{x} = f(x) + b_1 u_1$. Suppose b_1 is given by

$$b_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where $a,b,c \in \mathbb{R}$. Then the accessibility distribution $\mathcal{C}(0)$ is computed as

$$\mathcal{C} = \begin{bmatrix} a & -\frac{2\,b\,c\,(I_{22} - I_{33})}{I_{11}} & -\frac{2\,a\,c^2\,(I_{11} - I_{33})(I_{22} - I_{33})}{I_{11} I_{22}} \\ b & \frac{2\,a\,c\,(I_{11} - I_{33})}{I_{22}} & -\frac{2\,a^2\,b\,(I_{11} - I_{22})(I_{11} - I_{33})}{I_{22} I_{33}} - \frac{2\,b\,c^2\,(I_{11} - I_{33})(I_{22} - I_{33})}{I_{11} I_{22}} \\ c & -\frac{2\,a\,b\,(I_{11} - I_{22})}{I_{33}} & \frac{2\,b^2\,c\,(I_{11} - I_{22})(I_{22} - I_{33})}{I_{11} I_{33}} - \frac{2\,a^2\,c\,(I_{11} - I_{22})(I_{11} - I_{33})}{I_{22} I_{33}} \end{bmatrix}$$

Having obtained three independent directions, we may now conclude that the system is accessible. Let us further make the assumption that $I_{11} = I_{22}$, which implies that the spacecraft is symmetric. As discussed in section (2.1), the third axis of (2.14) is now constant. Then, $\mathcal{C}(0)$ becomes

$$\mathcal{C}(0) = \begin{bmatrix} a & -\frac{2\,b\,c\,(I_{22} - I_{33})}{I_{11}} & -\frac{2\,a\,c^2\,(I_{11} - I_{33})(I_{22} - I_{33})}{I_{11} I_{22}} \\ b & \frac{2\,a\,c\,(I_{11} - I_{33})}{I_{22}} & -\frac{2\,b\,c^2\,(I_{11} - I_{33})(I_{22} - I_{33})}{I_{11} I_{22}} \\ c & 0 & 0 \end{bmatrix} \tag{2.41}$$

To study the rank condition, we look at the invertibility condition of (2.41). The determinant of (2.41) is given by

$$\det(\mathcal{C}(0)) = \left(\frac{I_1 - I_3}{I_1}\right)^3 c^4 (a^2 + b^2)$$

Hence, (2.41) becomes singular only when either $I_{11} = I_{33}$, or $c = 0$, or $a = b = 0$. Suppose that $\det(\mathcal{C}(0)) \neq 0$, then (2.41) is invertible and $\dim \det(\mathcal{C}(0)) = 3$. Therefore, the assumed system of case III is accessible.

To show controllability of cases I-III, we invoke theorems 1 and 2 of Crouch [30]. By theorem 2 of [30], we have that the vector field f of system (2.35) is Poisson stable. Then, by theorem 1 of [30], we have that systems in Cases I-III are controllable when $\dim \mathcal{C} = 3$. In summary, we have shown the cases that given a bounded control, and the Poisson stable vector field f of (2.35), the system can be made accessible, and hence controllable. It should also be mentioned that for a linear system, the accessibility distribution reduces to the controllability matrix of section 2.4.

CHAPTER 3
Optimal Control

3.1 *Deterministic Optimal Control*

Consider the general deterministic dynamical system

$$\dot{x}_s = G(x_s, u(s), s) \quad t \leq s \leq T$$
$$x_t = x_o \in \mathbb{R}^n \qquad u(s) \in \mathbb{R}^m \tag{3.1}$$

where x_o is the initial condition of the differential equation. We are interested in calculating a control $u(s)$ that takes $x_t = x_o$ to a terminal state x_T, while minimizing an objective function. Such a task is known as an optimal control problem. In general, there exist two methods of solving an optimal control problem:

1) Dynamic Programing

2) Calculus of Variation (Pontryagin's Maximum Principle)

In this thesis, we will primarily focus on the first method: Dynamic Programing [37], developed by Richard Bellman in the 1950s. However, we will also use calculus of variations to derive the first-order necessary condition for optimality, and what is known as the Euler-Lagrange equation [32]. Given the dynamical system (3.1), we would like to minimize the cost functional

$$J_{x_o,t}[u(.)] = \int_t^T r(x_s, u(s), s)\, ds + \phi(x_T, T) \tag{3.2}$$

where the smooth and convex function $r : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, is called the *running cost*, and $\phi(x_T, T) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, is the *terminal cost*. Note that T may not be fixed. Most commonly, there are three types of problems with their respective cost functional forms [32]:

1) *Mayer problem*: when the cost functional is only a terminal cost.

2) *Bolza problem*: when both the terminal and running costs are present.

3) *Lagrange problem*: when $\phi(x_T, T) = 0$, and the cost functional becomes $J_{x_o,t}[u(.)] = \int_t^T r(x_s, u(s), s)\, ds$. A special case of the Lagrange problem, the *infinite horizon problem*, is often considered when $T \to \infty$. We will use the infinite horizon setting in the incoming sections to derive the optimal control for spacecraft attitude dynamics.

To start using calculus of variation we derive the differential equations which their solution will minimize the cost functional J. Such equations are known as the first order necessary conditions of optimality. Introducing the Lagrange multipliers $\lambda \in \mathbb{R}^n$, the cost functional with the dynamic constraint is rewritten as $\hat{J}$

$$\hat{J}[u(.)] = \phi(x_T, T) + \int_t^T r(x_t, u(t), t) + G(x_t, u(t)) - \dot{x}_t)^T \lambda(t)\, dt$$

Let us now define the control Hamiltonian, $H: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, as

$$H(x_t, u(t), \lambda(t), t) = r(x_t, u(t), t) + G(x_t, u(t))^T \lambda(t)$$

Substituting H into the cost functional, fixing the initial and terminal times t_o and t_f, and integrating the $-(\dot{x})^T \lambda(t)$ term by parts [38], $\hat{J}$ becomes

$$\hat{J} = \phi\left(x_{t_f}, t_f\right) + \left(x_{t_o}\right)^T \lambda(t_o) - \left(x_{t_f}\right)^T \lambda\left(t_f\right) + \int_{t_o}^{t_f} H(x_t, u(t), \lambda(t), t) + (x_t)^T \dot{\lambda}(t)\ dt$$

By introducing variation in $u(t)$, the δu, we'll cause variation in x_t, δx, and ultimately variation in $\hat{J}$, $\delta \hat{J}$. Hence, the cost function becomes

$$\delta \hat{J} = \left[(\delta x)^T \left(\frac{\partial \phi}{\partial x} - \lambda\right)\right]_{t=t_f} + [(\delta x)^T \lambda]_{t=t_o} + \int_{t_o}^{t_f} (\delta x)^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}\right) + (\delta u)^T \frac{\partial H}{\partial u}\ dt \qquad (3.3)$$

Since the Lagrange multipliers are arbitrarily introduced to force the constraint into the objective function, and since at the optimal point, variation of the cost functional must be zero, the Lagrange multipliers are chosen in a way to make the coefficients of variations δx to go to zero. Hence, let us choose $\dot{\lambda}$ as

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{3.4}$$

The integral term of (3.3) becomes $\int_{t_o}^{t_f} (\delta u)^T \frac{\partial H}{\partial u} \, dt$, and the boundary condition follows as

$$\lambda(t_f) = \left[\frac{\partial \phi}{\partial x}\right]_{t=t_f} \tag{3.5}$$

In particular by distributing the differentiation in (3.4), we have

$$\dot{\lambda} = -\left(\frac{\partial r}{\partial x}\right) - \left(\frac{\partial G}{\partial x}\right)^T \lambda$$

Equation (3.4) is known as the *costate* equation [38]. Inputting $\dot{\lambda}$ and $\lambda(t_f)$ into (3.3), the cost functional variation becomes

$$\delta \hat{J} = [(\delta x)^T \lambda]_{t=t_o} + \int_{t_o}^{t_f} (\delta u)^T \frac{\partial H}{\partial u} \, dt$$

Moreover, we have that at the optimal point, variation of the cost functional must be zero. Hence, we have that $\delta \hat{J} = 0$. This implies that

$$\frac{\partial H}{\partial u} = 0 \tag{3.6}$$

All in all, to find an optimal control that minimizes the cost functional J, following set of equations need to be solved

$$\begin{gathered} \dot{x}_t = G(x_t, u(t), t) \\ \dot{\lambda} = -\frac{\partial r}{\partial x} - \left(\frac{\partial G}{\partial x}\right)^T \lambda = -\frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial u} = \frac{\partial r}{\partial u} + \left(\frac{\partial G}{\partial u}\right)^T \lambda = 0 \end{gathered} \tag{3.7}$$

with boundary conditions

$$x_{t=t_o} = x_o$$

$$\lambda(t_f) = \left[\frac{\partial \phi}{\partial x}\right]_{t=t_f}$$

The third and second equations of (3.7), along with the second boundary condition are known as the *Euler-Lagrange* equations [38]. The two boundary conditions are also known as the *transversality* conditions. The costate equation is solved backward in time and satisfies the terminal condition $\lambda(t_f)$, while the state equation is solved forward in time and satisfies the initial condition $x_{t=t_o}$. This is known as a *two-point boundary value* problem. There are several methods that provide an estimate of the optimal control $u(t)$ (i.e. based on an initial guess) to minimize the cost functional. In certain special cases, and when the dynamics are linear, these equations are easier to solve. However, in most practical applications, and physical problems, deriving an exact solution is difficult. Thus, numerical methods are often utilized to obtain the approximate optimal control. Several of these methods have been discussed in Betts [39].

3.2 *Dynamic Programing*

In this section, we would like to derive the HJB partial differential equation. The HJB equation is of interest since the solution to the HJB gives the optimal cost of the optimal control problem. The steps shown in this section follow the dynamic programing derivations of Nemhauser [40]. Consider the dynamical system (3.1), along with the cost functional (3.2) for $0 \leq t \leq T$, and $x_o \in \mathbb{R}^n$. The goal of dynamic programing is to find $u^*(.) \in \mathbb{U}$, the optimal control for the dynamics (3.1) in the time interval $[0, T]$, such that the relation

$$J[u^*(.)] \leq J[u(.)],\ \forall\, u(.) \in \mathbb{U}.$$

is satisfied [32]. Let $u^*(.)$ be the optimal control and $x^*(.)$ the corresponding trajectory of the controlled dynamics. By choosing $t \in [0, T]$, we will denote the corresponding state of the optimally controlled plant at time t by x_t^*. Then $u^*(.)$ restricted to $[t,T]$ must be optimal for the following optimization problem:

$$\text{minimize} \left[\int_t^T r(x_s, u(s), s)\, ds + \phi(x_T, T)\right]$$
$$\text{subject to} \quad \dot{x}_s = G(x_s, u(s), s), \quad t \le s \le T \tag{3.8}$$

Specifically, if we were able to find the optimal control trajectory in the interval $[0,T]$ by solving the optimal control problem, then the resulting $x^*(.)$ is also optimal on all the subintervals of the form $[t,T] \subset [0,T]$ with $t > 0$. This is provided that the initial condition x_t^* at time t is obtained from running the system forward along the optimal trajectory from time $t = 0$ [32].

Moreover, due to *Bellman's principle of optimality* [32] we have that if some other control $u(.)^{**}$ on $[t,T]$ achieved a strictly lower cost, then the concatenation of $u(.)^*$ on $[0,t]$ and $u(.)^{**}$ on $[t,T]$ will yield a cost over the entire interval $[0,T]$ which is strictly less than that achieved by $u^*(t)$. Let us now use this principle to derive the HJB PDE associated with (3.1).

Consider the following Lagrange problem (i.e. special case of (3.8) where $\phi(x_T, T) = 0$)

$$\text{minimize} \left[\int_t^T r(x_t, u(t), t)\, dt\right]$$

$$\text{subject to } \dot{x} = G(x_t, u(t), t), \quad x_t = x_o$$

To formulate the problem as a dynamic program, we will define the *value function*, $V(x)$: $\mathbb{R}^n \to \mathbb{R}$ such that

$$V(x_t, t) = \min_{\substack{u(t) \\ [t,T]}} \int_t^T r(x_t, u(t), t)\, dt \tag{3.9}$$

The value function is the best value (or cost) of the objective. By additivity property of integrals, value function can be written as a summation of two integrals

$$V(x(t), t) = \min_{\substack{u(t) \\ [t,T]}} \left(\int_{t+\delta t}^T r(x_t, u(t), t)\, dt + \int_t^{t+\delta t} r(x_t, u(t), t)\, dt \right)$$

where $[t,T] \rightarrow [t, t+\delta t] \cup [t+\delta t, T]$. Further dividing the minimization interval into two intervals, we have that

$$V(x(t),t) = \min_{\substack{u(t) \\ [t,t+\delta t]}} \left(\min_{\substack{u(t) \\ [t+\delta t,T]}} \left(\int_{t+\delta t}^{T} r(x_t, u(t), t)\, dt + \int_{t}^{t+\delta t} r(x_t, u(t), t)\, dt \right) \right)$$

Realizing that the second integral is only over the interval $[t+\delta t, T]$,we obtain the following form of the value function

$$V(x(t),t) = \min_{\substack{u(t) \\ [t,t+\delta t]}} \left(\int_{t}^{t+\delta t} r(x_t, u(t), t)\, dt + \min_{\substack{u(t) \\ [t+\delta t,T]}} \int_{t+\delta t}^{T} r(x_t, u(t), t)\, dt \right) \tag{3.10}$$

By definition, we have that

$$V\big(x_{(t+\delta t)}, t+\delta t\big) = \min_{\substack{u(t) \\ [t+\delta t,T]}} \int_{t+\delta t}^{T} r(x_t, u(t), t)\, dt \tag{3.11}$$

Substituting (3.11) in equation (3.10), we obtain the following expression

$$V(x(t),t) = \min_{\substack{u(t) \\ [t,t+\delta t]}} \left(\int_{t}^{t+\delta t} r(x_t, u(t), t)\, dt + V\big(x_{(t+\delta t)}, t+\delta t\big) \right) \tag{3.12}$$

where the $\int_{t}^{t+\delta t} r(x_t, u(t), t)\, dt$ term is the immediate return, and $V\big(x_{(t+\delta t)}, t+\delta t\big)$ term is known as the optimal return on $[t+\delta t, T]$. For sufficiently small δt, the cost function becomes

$$\int_{t}^{t+\delta t} r(x_t, u(t), t)\, dt = r(x, u, t)\, \delta t$$

Hence, (3.12) can be restated as

$$V(x(t),t) = \min_{\substack{u(t) \\ [t,t+\delta t]}} \big(r(x,u,t)\delta t + V(x(t+\delta t), t+\delta t) \big) \tag{3.13}$$

Next, consider the Taylor series expansion of function $V(t)$ around the constant $c \in \mathbb{R}$

$$V(t) = V(c) + \left(\frac{dV(c)}{dt}\right)(t-c) + \left(\frac{d^2V(c)}{dt^2}\right)\frac{(t-c)^2}{2!} + \cdots$$

and recall that $\left(\frac{dV(c)}{dt}\right)$ can be written as

$$\left(\frac{dV(c)}{dt}\right) = \left(\frac{dV(x_t,t)}{dt}\right) = \frac{\partial V(x_t,t)}{\partial t} + \dot{x}_t^{\,T}\, \frac{\partial V(x_t,t)}{\partial x}$$

where $\dot{x}_t = \left(\frac{dx_t}{dt}\right)$ is the closed loop dynamics (3.1). Using the above Taylor series expansion, we expand $V(x_{(t+\delta t)}, t+\delta t)$ around $c = (x_t, t)$ such that

$$V\left(x_{(t+\delta t)}, t+\delta t\right) = V(x_t, t) + \left(\frac{dV(c)}{dt}\right)(t+\delta t - t)$$

$$\Rightarrow V\left(x_{(t+\delta t)}, t+\delta t\right) = V(x_t, t) + \left(\frac{dV(c)}{dt}\right)\delta t$$

$$\Rightarrow V\left(x_{(t+\delta t)}, t+\delta t\right) = V(x_t, t) + \left(\frac{\partial V(x_t, t)}{\partial t} + \dot{x}_t{}^T \frac{\partial V(x_t, t)}{\partial x}\right)\delta t$$

Substituting the obtained expression in (3.13), we obtain

$$V(x_t, t) = \min_{\substack{u(t) \\ [t, t+\delta t]}} \left(r(x, u, t)\delta t + V(x_t, t) + \left(\frac{\partial V(x_t, t)}{\partial t} + \dot{x}_t{}^T \frac{\partial V(x_t, t)}{\partial x}\right)\delta t \right)$$

Note that we have ignored the higher order terms of the Taylor expansion, as well as the δt^2 terms. Additionally, since $V(x_t, t)$ does not depend on the minimization variable $u(t)$, we shall subtract the $V(x_t, t)$ term from both sides of the equality such that

$$0 = \min_{\substack{u(t) \\ [t, t+\delta t]}} \left(r(x, u, t)\delta t + \left(\frac{\partial V(x_t, t)}{\partial t} + \dot{x}_t{}^T \frac{\partial V(x_t, t)}{\partial x}\right)\delta t \right) \tag{3.14}$$

Further dividing by δt, equation (3.14) becomes

$$0 = \min_{\substack{u(t) \\ [t, t+\delta t]}} \left(r(x, u, t) + \frac{\partial V(x_t, t)}{\partial t} + \dot{x}_t{}^T \frac{\partial V(x_t, t)}{\partial x} \right)$$

Finally, taking the limit $\delta t \rightarrow 0$, and substituting the closed loop dynamics (3.1), we arrive at the following partial differential equation:

$$0 = \frac{\partial V(x_t, t)}{\partial t} + \min_{u(t)} \left(r(x, u, t) + \sum_{i=1}^{n} G_i(x_t, u(t), t) \frac{\partial V(x_t, t)}{\partial x_i} \right) \tag{3.15}$$

Equation (3.15) is the *Hamilton-Jacobi-Bellman* (HJB) equation. It is of great importance to us, since the solution to the HJB is the value function, which is the optimal cost of the given optimal control problem. Moreover, if the value function is obtained, we can always find the optimal control that minimizes the cost functional. In the consequent chapters, we formulate the HJB for our physical problem of interest. However, we may first make an additional assumption: we let $T \to \infty$ so that the system settles into a steady state. As a result, we have that the value function does not explicitly depend on time, i.e. $\frac{\partial V(x_t)}{\partial t} = 0$. Hence (3.15) simplifies to

$$0 = \min_{u(t)} \left(r(x,u) + \sum_{i=1}^{n} G_i\big(x_t, u(t)\big) \frac{\partial V(x_t)}{\partial x_i} \right) \tag{3.16}$$

Furthermore, differentiating the HJB PDE with respect to u, We obtain the equation

$$0 = \frac{r(x,u)}{\partial u} + \sum_{i=1}^{n} \frac{G_i\big(x_t, u(t)\big)}{\partial u} \frac{\partial V(x_t)}{\partial x_i} \tag{3.17}$$

which is a necessary condition for a minimum. Recall the optimality condition (3.6) of the previous section. Clearly, the obtained condition (3.17) is the same as condition (3.6). It can also be observed that in the Hamiltonian formulation, the gradient of value function is the Lagrange multiplier. Consider the following Hamiltonian

$$H(x,u,\lambda) = r(x,u) + G(x_t,u)^T \lambda$$

Substituting $\lambda = \frac{\partial V(x_t)}{\partial x}$ for the Lagrange multiplier, we obtain

$$H(x,u) = r(x,u) + G(x_t,u)^T \frac{\partial V(x_t)}{\partial x}$$

which is the familiar form of the HJB equation. Then, the Hamiltonian form of the HJB equation can be written as

$$0 = \min_{u \in U} \left\{ H\left(x, u, \frac{\partial V(x_t)}{\partial x}\right) \right\}$$

In section 3.4, the HJB corresponding to a stochastic system is derived.

3.3 *Itô's Lemma and the Diffusion Generator*

In this section, we will derive the identity of Itô lemma. This will enable us to express the differential of a function of stochastic process. We will further derive the infinitesimal generator for a SDE. Consider the following form of the Itô stochastic differential equation

$$X_T = X_{t_o} + \int_{t_o}^{T} G(X_t)\, dt + \int_{t_0}^{T} \sigma(X_t)\, dW_t$$
$$dX_t = G(X_t)dt + \sigma(X_t)dW_t, \quad t_o < t < T \tag{3.18}$$

where W is a one-dimensional Brownian motion, $X \in \mathbb{R}$, and $G: \mathbb{R} \to \mathbb{R}$, and $t_o \geq 0$. We would like to arrive at the expression known as the Itô lemma through Taylor expansion of a measurable function $\psi: \mathbb{R} \to \mathbb{R}$, where ψ is at least C^2. For any $t \in [t_o, T]$, consider the Itô-Taylor expansion of the function ψ around X_t

$$\psi(X_{t+\delta t}) = \psi(X_t) + (X_{t+\delta t} - X_t)\frac{\partial\psi(X_t)}{\partial x} + \frac{1}{2}(X_{t+\delta t} - X_t)^2\frac{\partial^2\psi(X_t)}{\partial x^2} + \mathcal{O}((X_{t+\delta t} - X_t)^3) \tag{3.19}$$

Here by $\frac{\partial\psi(X_t)}{\partial x}$, we mean $\left[\frac{\partial\psi(x)}{\partial x}\right]\Big|_{x=X_t}$ which is the derivative of $\psi(X_t)$ with respect to the state argument evaluated at location X_t. Realizing that $(X_{t+\delta t} - X_t)$ is given by the SDE (3.18) as $G(X_t)\delta t + \sigma(X_t)\delta W_t$, we substitute for the $(X_{t+\delta t} - X_t)$, and the $(X_{t+\delta t} - X_t)^2$ terms

$$\psi(X_{t+\delta t}) = \psi(X_t) + (G(X_t)\delta t + \sigma(X_t)\delta W_t)\frac{\partial\psi(X_t)}{\partial x} + \frac{1}{2}\left(G(X_t)^2\delta t^2 + 2G(X_t)\sigma(X_t)\delta t\delta W_t + \sigma(X_t)^2\left(\delta W_t\right)^2\right)\frac{\partial^2\psi(X_t)}{\partial x^2} + \mathcal{O}((X_{t+\delta t} - X_t)^3) \tag{3.20}$$

Inspecting the higher order terms in δt, we see that $\delta t\delta W_t \sim \delta t^{3/2}$, and $(\delta W_t)^2 \sim \delta t$. Consequently, all higher order terms in δt which are close to zero are neglected [41]. Additionally, $(\delta W_t)^2$ is approximated by δt. Using these approximations, (3.20) becomes

$$\psi(X_{t+\delta t}) = \psi(X_t) + (G(X_t)\delta t + \sigma(X_t)\delta W_t)\frac{\partial\psi(X_t)}{\partial x} + \frac{1}{2}(\sigma(X_t)^2\delta t)\frac{\partial^2\psi(X_t)}{\partial x^2} + \mathcal{O}(\delta t^{3/2}) \tag{3.21}$$

Rearranging the $\psi(X_t)$ term in (3.21), substituting back $(X_{t+\delta t} - X_t) = dX_t$, and ignoring the higher order error, we have

$$d\psi(X_t) = \frac{\partial\psi(X_t)}{\partial x}dX_t + \frac{1}{2}\sigma(X_t)^2\frac{\partial^2\psi(X_t)}{\partial x^2}dt \tag{3.22}$$

Expression (3.22) is known as the *Itô's lemma* and describes the derivative of a time-independent function of stochastic process. We will also derive the time-dependent case of this well-known equation in this section. Let us now define the generator of Itô diffusion

$$\mathcal{L}\psi(x) = \lim_{\delta t\to 0}\frac{\mathbb{E}[\psi(X_{t+\delta t})|X_t = x] - \psi(x)}{\delta t} = \lim_{\delta t\to 0}\frac{\mathbb{E}[\psi(X_{t+\delta t}) - \psi(x)|X_t = x]}{\delta t} \tag{3.23}$$

From (3.21), the term

$$\psi(X_{t+\delta t}) - \psi(X_t) = (G(X_t)\delta t + \sigma(X_t)\delta W_t)\frac{\partial\psi(X_t)}{\partial x} + \frac{1}{2}(\sigma(X_t)^2\delta t)\frac{\partial^2\psi(X_t)}{\partial x^2}$$

is substituted in for (3.23), thus evaluating the limit, we have

$$\mathcal{L}\psi(x) = G(X_t)\frac{\partial\psi(X_t)}{\partial x} + \frac{1}{2}\sigma(X_t)^2\frac{\partial^2\psi(X_t)}{\partial x^2} \tag{3.24}$$

This expression is known as the Generator of Itô diffusion, or, the infinitesimal generator of stochastic process [41]. The first term of the generator is known as the drift term, and the second term is known as the diffusion term. These results can be extended to higher dimensions, for the n-dimensional $\psi: \mathbb{R}^n \to \mathbb{R}$, see, for example Øksendal [42], chapters 4 and 7. Suppose, $X \in \mathbb{R}^n$, and W is m-dimensional Brownian motion, then the infinitesimal generator of the diffusion associated with the SDE (3.18) is given by

$$\mathcal{L}\psi(x) = \sum_i G_i(X_t)\frac{\partial\psi(X_t)}{\partial x_i} + \frac{1}{2}\sum_{i,j} a_{i,j}(X_t)\frac{\partial^2\psi(X_t)}{\partial x_i\partial x_j} \tag{3.25}$$

where, $a_{i,j}(X) = (\sigma(X)\sigma(X)^T)_{i,j}$. We will use (3.25) in the latter sections of this thesis.

Now we would like to obtain a general form of the Itô's Lemma where the function ψ is time-dependent. Recall the Itô's lemma (3.22) for the time independent function $\psi(X_t)$. Replacing $\psi(X_t)$ with the time dependent $\psi(X_t,t)$, and adding the term accounting for the time variation of $\psi(X_t,t)$, i.e. $d\psi(X_t,t) = \left(\frac{\partial\psi(X_t,t)}{\partial x}dX_t + \frac{\partial\psi(X_t,t)}{\partial t}dt\right)$, we have

$$d\psi(X_t,t) = \frac{\partial\psi(X_t,t)}{\partial x}dX_t + \frac{1}{2}\sigma(X_t)^2\frac{\partial^2\psi(X_t,t)}{\partial x^2}dt + \frac{\partial\psi(X_t,t)}{\partial t}dt \tag{3.26}$$

Realizing that, $dX_t = G(X_t)dt + \sigma(X_t)dW_t$, and substituting for dX_t in (3.26), we obtain

$$d\psi(X_t,t) =$$

$$\frac{\partial\psi(X_t,t)}{\partial x}(G(X_t)dt + \sigma(X_t)dW_t) + \frac{1}{2}\sigma(X_t)^2\frac{\partial^2\psi(X_t,t)}{\partial x^2}dt + \frac{\partial\psi(X_t,t)}{\partial t}dt$$

Simplifying and rearranging further, we have

$$d\psi(X_t,t) =$$

$$\frac{\partial\psi(X_t,t)}{\partial t}dt + \left[G(X_t)\frac{\partial\psi(X_t,t)}{\partial x} + \frac{1}{2}\sigma(X_t)^2\frac{\partial^2\psi(X_t,t)}{\partial x^2}\right]dt + \sigma(X_t)\frac{\partial\psi(X_t,t)}{\partial x}dW_t \tag{3.27}$$

The terms in the brackets are recognized as the infinitesimal generator of stochastic process (3.25), and are written as $\mathcal{L}\psi(X_t,t) = G(X_t)\frac{\partial\psi(X_t,t)}{\partial x} + \frac{1}{2}\sigma(X_t)^2\frac{\partial^2\psi(X_t,t)}{\partial x^2}$, thus (3.27) becomes

$$d\psi(X_t,t) = \frac{\partial\psi(X_t,t)}{\partial t}dt + [\mathcal{L}\psi(X_t,t)\,dt] + \sigma(X_t)\frac{\partial\psi(X_t,t)}{\partial x}dW_t \tag{3.28}$$

More generally, in integral form we have

$$\psi(X_T,T) = \psi\left(X_{t_o},t_o\right) + \int_{t_o}^{T}\left(\frac{\partial\psi(X_t,t)}{\partial t} + \mathcal{L}\psi(X_t,t)\right)dt + \int_{t_o}^{T}\sigma(X_t)\frac{\partial\psi(X_t,t)}{\partial x}dW_t \tag{3.29}$$

Hence, we have obtained the Itô Lemma for a time-dependent function ψ. Similarly, here extension to the multidimensional case is straight forward, and we may assume that $X \in \mathbb{R}^n$.

3.4 *Stochastic Dynamic Programming*

In stochastic control, disturbances are modeled as random processes (as shown in section 2.3), and the performance index is the average over all the sample paths of the solution to the stochastic differential equation [32]. In this section, we will derive the HJB PDE for a stochastic system. Recall the Itô SDE

$$dX_s = G(X_s, u(s))\, ds + \sigma(X_s) dW_s, \quad t \le s \le T \\ X_t = x \in \mathbb{R}^n, \tag{3.30}$$

Let the expected cost functional be

$$J_{x,t}[u(.)] = \mathbb{E}_{x,t}\left[\int_t^T r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right] \tag{3.31}$$

We are interested in the above's problem for all choices of initial times $0 \le t \le T$, and all choices of initial states x [32]. Hence, we define the value function starting at point x at time t as

$$V(x,t) = \min_{u(.)} J_{x,t}[u(.)] = \min_{u(.)} \mathbb{E}_{x,t}\left[\int_t^T r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right] \tag{3.32}$$

where at terminal time $t = T$, $V(x,T) = \phi(x)$. By Bellman's principle of optimality (see section 3.2), we restate (3.31) for any control $u(.)$ as

$$V(x,t) \le \mathbb{E}_{x,t}\left[\int_t^T r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right] \tag{3.33}$$

By additivity property of integrals, (3.33) becomes

$$V(x,t) \le \mathbb{E}_{x,t}\left[\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds + \int_{t+\delta t}^T r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right] \tag{3.34}$$

Next, consider the property $\mathbb{E}_{x_o}[\psi(X_s)] = \mathbb{E}_x[\psi(X_s)|X_o = x_o]$ for $s \ge 0$ and a measurable function ψ. Setting $T = s + t$, and $t_o = t$, we have $\mathbb{E}_{x,t}[\psi(X_T)] = \mathbb{E}_x[\psi(X_T)|X_t = x]$. Further using the Tower Property [32], this expression is restated as

$$\mathbb{E}_{x,t}[\psi(X_T)|X_t = x] = \mathbb{E}[\mathbb{E}[\psi(X_T)|X_{t+\delta t}] \mid X_t = x]$$

where $\delta t > 0$. Then using the Markov Property [32], we have

$$\mathbb{E}[\mathbb{E}[\psi(X_T)|X_{t+\delta t}] \mid X_t = x] = \mathbb{E}\left[\mathbb{E}_{X_{t+\delta t},t+\delta t}[\psi(X_T)] \mid X_t = x\right] = \mathbb{E}_{x,t}\left[\mathbb{E}_{X_{t+\delta t},t+\delta t}[\psi(X_T)]\right]$$

Therefore, we've obtained the property

$$\mathbb{E}_{x,t}[\psi(X_T)] = \mathbb{E}_{x,t}\left[\mathbb{E}_{X_{t+\delta t},t+\delta t}[\psi(X_T)]\right] \tag{3.35}$$

Using the property (3.35) on expression (3.34), we have

$$\mathbb{E}_{x,t}\left[\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds + \int_{t+\delta t}^{T} r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right] =$$

$$\mathbb{E}_{x,t}\left[\mathbb{E}_{X_{t+\delta t},t+\delta t}\left[\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds + \int_{t+\delta t}^{T} r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right]\right]$$

$$\Rightarrow\; V(x,t) \le \mathbb{E}_{x,t}\left[\mathbb{E}_{X_{t+\delta t},t+\delta t}\left[\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds + \int_{t+\delta t}^{T} r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right]\right]$$

Note that the term $\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds$ is not in the interval $[t + \delta t, T]$, thus, we write

$$V(x,t) \le \mathbb{E}_{x,t}\left[\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds + \mathbb{E}_{X_{t+\delta t},t+\delta t}\left[\int_{t+\delta t}^{T} r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right]\right] \tag{3.36}$$

Next, assume that the control applied from t to $t + \delta t$ is arbitrary, and that the control applied in the interval $[t + \delta t]$ (the second integral) is optimal. Thus, we have

$$\mathbb{E}_{X_{t+\delta t},t+\delta t}\left[\int_{t+\delta t}^{T} r\big(X_s, u(s)\big)\, ds + \phi(X_T)\right] = V(X_{t+\delta t}, t + \delta t) \tag{3.37}$$

Substituting (3.37) and the inequality becomes

$$V(x,t) \le \mathbb{E}_{x,t}\left[\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds + V(X_{t+\delta t}, t + \delta t)\right]$$

Note that, unlike the $V(x,t)$ term, the $V(X_{t+\delta t}, t + \delta t)$ term is random with respect to the expectation $\mathbb{E}_{x,t}$ [32]. Hence, the inequality is restated as

$$0 \leq \mathbb{E}_{x,t}\left[\int_t^{t+\delta t} r\big(X_s, u(s)\big)\, ds + V(X_{t+\delta t}, t+\delta t) - V(x,t)\right] \tag{3.38}$$

Recall the Itô lemma (3.29) from the previous section. Applying the Itô lemma to $V(X_{t+\delta t}, t+\delta t)$, we have

$$V(X_{t+\delta t}, t+\delta t) - V(x,t) =$$
$$\int_t^{t+\delta t}\left(\frac{\partial}{\partial s}V(X_s,s) + \mathcal{L}V(X_s,s)\right)ds + \int_t^{t+\delta t} \nabla V^T(X_s,s)\sigma(X_s,s)\, dW_s \tag{3.39}$$

Substituting (3.39) into (3.38), we further obtain

$$0 \leq \mathbb{E}_{x,t}\left[\int_t^{t+\delta t}\left(r\big(X_s,u(s)\big) + \frac{\partial}{\partial s}V(X_s,s) + \mathcal{L}V(X_s,s)\right)ds\right] \tag{3.40}$$

Note that the expected value of the term with Brownian motion is zero, thus, (3.40) does not include this term. Let us now divide the integral by δt, and take the limit of the expectation as $\delta t \to 0$. We have

$$0 \leq \lim_{\delta t \to 0} \mathbb{E}_{x,t}\left[\frac{1}{\delta t}\int_t^{t+\delta t}\left(r\big(X_s,u(s)\big) + \frac{\partial}{\partial s}V(X_s,s) + \mathcal{L}V(X_s,s)\right)ds\right]$$

Evaluating the limit results in terms given at time t, i.e. $X_t = x$, and $u(t) = u$ [32], thus the obtained expression becomes deterministic:

$$0 \leq r(x,u) + \frac{\partial}{\partial t}V(x,t) + \mathcal{L}V(x,t)$$

It is important here to mention that the generator term may depend on the control u as well. This will be due to terms such as $G(x,u)$, and $\sigma(x,u)$. Assuming that we have the optimal control u^* as the applied control, then the inequality becomes an equality. Specifically, we have

$$0 = \min_u\left(r(x,u) + \frac{\partial}{\partial t}V(x,t) + \mathcal{L}V(x,t)\right) \tag{3.41}$$

We have now arrived at the HJB equation associated with a stochastic differential equation. Rewriting (3.41) using the infinitesimal generator of stochastic process (3.25), and separating the terms that do not depend on the control u, we can write the HJB equation as

$$-\frac{\partial V(x,t)}{\partial t} = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j} + \min_u\left(r(x,u) + \sum_i G_i(x,u)\frac{\partial V(x,t)}{\partial x_i}\right) \tag{3.42}$$

and, assuming that u^* is found and is optimal, we obtain the following form of HJB PDE

$$-\frac{\partial V(x,t)}{\partial t} = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j} + r(x,u^*) + \sum_i G_i(x,u^*)\frac{\partial V(x,t)}{\partial x_i}$$
$$V(x,T) = \phi(x) \tag{3.43}$$

In particular, by applying the Feynman-Kac formula [32] we have

$$V(x,t) = \mathbb{E}_{x,t}\left[\int_t^T r(X_s^*, u^*(X_s^*, s))\, ds + \phi(X_T^*)\right]$$

where the optimal trajectory, X_t^*, is the solution to the SDE (3.30), such that

$$dX_s^* = G(X_s^*, u^*(X_s^*, s))\, ds + \sigma(X_s^*)dW_s, \quad t \le s \le T$$

In comparison to equation (3.15), we see that the difference between the HJB PDE (3.42) associated with the SDE (3.30) and a HJB PDE associated with deterministic constraint is the term $\frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}V(x,t)$ which is the diffusion term of the infinitesimal generator. Similarly, here we may make the assumption that $T \to \infty$, such that the system settles into steady state. As a result, we have that the value function will no longer be a function, and therefore $\frac{\partial}{\partial t}V(x) = 0$. At steady state, the HJB becomes

$$0 = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 V(x)}{\partial x_i \partial x_j} + \min_u\left(r(x,u) + \sum_i G_i(x,u)\frac{\partial V(x)}{\partial x_i}\right) \tag{3.44}$$

It is possible to solve the HJB PDE in exact form when, for instance, the dynamics $G_i(x,u)$ are linear, and the objective function is quadratic. However, the problem could become difficult in certain cases with nonlinear dynamics. Hence similarly here, approximations would be needed to obtain the approximate optimal control and the solution to the HJB. In this study, a specific form of the HJB PDE associated with the SDE (2.26) has arisen in the application of spacecraft attitude dynamics. We are specifically interested in the HJB PDE with nonlinear dynamics and multiplicative linear control in the diffusion term. For specifics of the modeling of this problem, reader may refer to section 2.3 of this thesis. Due to the application, the HJB PDE of interest is

$$0 = \min_u \left(r(x,u) + \frac{1}{2}\sum_{i,j} a_{i,j}(u)\frac{\partial^2}{\partial x_i \partial x_j} V(x,t) + \sum_i G_i(x,u)\frac{\partial}{\partial x_i} V(x,t) \right)$$

where $a_{i,j}(u) = \sigma(u)\sigma^2(u)$. In the next chapter, we will outline the powerful method of Al'brekht [1] which utilizes the power series expansion of different terms of the HJB equation to provide an approximate solution to the HJB locally. This specific method is of interest because of its efficiency in solving nonlinear problems, as well as its ability to approximate the solution to a degree of approximation determined by the user. We will discuss the numerical and computational complications associated with this method in the latter sections.

CHAPTER 4
The Al'brekht Method

4.1 *Al'brekht Method*

In this chapter, a brief overview of the method known as the *Al'brekht method* [1] is discussed. Formulated originally by E. G. Al'brekht in 1960s, this method has been studied and extended in [43], [44], [45], [46], and [47]. Moreover, the method has been explored and utilized in many different applications and fields. The Al'brekht method is concerned with providing an approximate analytic optimal control for stabilization of a nonlinear system. Concisely, Al'brekht's approach is the expansion of power series of the value function, control, dynamics, and the running cost, and substitution of the truncated expansions into the HJB equation. Then, by grouping the HJB equation at different orders in x, the solution to the HJB (value function and the control) is obtained at every order. Consider the following nonlinear deterministic differential equation

$$\dot{x}_t = G(x_t, u(t)), \quad x(0) = x_o, \\ x \in \mathbb{R}^n,\ u \in \mathbb{R} \tag{4.1}$$

We are interested in minimizing the cost functional

$$J(u) = \int_0^\infty r(x_t, u(t))\, dt \tag{4.2}$$

in $0 \leq t \leq \infty$, through appropriate choice of function u. In this chapter, we will first outline the sufficient conditions of control optimality, which were originally given by Al'brekht [1]. Next, we will show a construction of optimal control following the method of Al'brekht, and provide the solvability conditions for higher orders of control for the deterministic system (4.1). Finally, we will briefly provide a discussion on convergence and error associated with the method.

4.2 *The Sufficient Condition of Optimal Control*

Al'brekht provided a sufficient condition for existence of stabilizing optimal control. His conditions used the results of Bellman in dynamic programing (See section 3.2), as well as, Lyapunov's stability argument. Specifically, he argued that if the control function $u(x)$, and a value function $V(x)$ are obtainable and satisfy the following three conditions, then the control $u = u(x)$ is the optimal control [1]. The sufficient conditions for optimality of control are as follows:

Condition I. The value function $V(x)$ must satisfy the Lyapunov asymptotic stability argument [1]. In fact, the value function $V(x): \mathbb{R}^n \to \mathbb{R}$, itself is the Lyapunov function candidate, hence, it must satisfy the conditions of the Lyapunov's second method for stability.

Specifically, consider the controlled system (4.1), and assume that it has an equilibrium at the origin. Suppose that $V(x)$ is a smooth positive definite function, i.e. we have that, $V(x) = 0$ if and only if $x = 0$, and $V(x) > 0$ if and only if $x \neq 0$. Then $V(x)$ is a Lyapunov function candidate, and system (4.1) is asymptotically stable if

$$\frac{dV(x)}{dt} = \sum_{i=1}^{n} G_i(x,u) \frac{\partial V(x)}{\partial x_i} < 0 \tag{4.3}$$

for all $x \neq 0$ [48]. Moreover, (4.1) is locally asymptotically stable for all x in the neighborhood of the equilibrium [49]. Note that the strict inequality is required for asymptotic stability of the system in a Lyapunov sense. However, due to Barbashin-Krasovskii-LaSalle principle, if $\dot{V}(x) \leq 0$, and the set $S = \{x | \frac{dV(x)}{dt} = 0\}$ does not contain any other trajectory of the system except $x(t) = 0$, $t \to \infty$, then the origin is said to be asymptoticly stable [50].

Condition II. Given the controlled system (4.1), the derivative of the value function, $\frac{dV(x)}{dt}$, must satisfy the equation

$$\frac{dV(x)}{dt} = -r(x, u(x)) \tag{4.4}$$

A closer look at this equation reveals that this equation is the steady-state HJB equation (3.16) when $u = u(x)$ is optimal. Further from condition I, we have that for the pair of functions $V(x)$, and $u(x)$ obtained from the HJB equation, (4.1) is asymptotically stable if

$$\sum_{i=1}^{n} G_i(x_t, u(x_t)) \frac{\partial V(x_t)}{\partial x_i} = -r(x_t, u(t)) \leq 0 \tag{4.5}$$

Condition III. The following function $H: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$H(x, u) = \frac{dV(x)}{dt} + r(x, u(x)) \tag{4.6}$$

must have a minimum at each point x in a neighborhood of the origin [1]. If so, then the control function $u(x)$ is the optimal control. That is assuming that the control Hamiltonian

$$H(\lambda, x, u) = G_i(x, u)^T \lambda + r(x, u)$$

is strictly convex in u, where $\lambda = \frac{\partial V(x)}{\partial x}$ is the Lagrange multiplier.

Suppose that the control $u^*(x)$, and the value function $V^*(x)$ satisfy the conditions I-III, then from the HJB equation (3.16), we have the following system of equations

$$\begin{aligned} 0 &= \sum_{i=1}^{n} G_i(x, u^*(x)) \frac{\partial V^*(x)}{\partial x_i} + r(x, u^*(x)) \\ 0 &= \sum_{i=1}^{n} \frac{G_i\big(x, u^*(x)\big)}{\partial u} \frac{\partial V^*(x)}{\partial x_i} + \frac{r\big(x, u^*(x)\big)}{\partial u} \end{aligned} \tag{4.7}$$

Note that the second equation of (4.7) is obtained through minimization of the first equation of (4.7) over the control variable u. The goal of Al'brekht method is to provide an approximation of the real analytic functions $V^*(x)$ and $u^*(x)$ that satisfy the system of equations (4.7) in a neighborhood of the origin.

If the partial sums of the series $V^*(x)$ and $u^*(x)$ are found such that conditions I-III are locally satisfied, then the approximate control $u^*(x)$ is the optimal control locally around the origin, hence minimizing (4.2). We will outline the construction of such partial sums, as in Al'brekht [1], in the next section. Reader may refer to Al'brekht [1] pages 1255-1256 for a formal proof of convergence of functions $V(x)$ and $u(x)$.

4.3 *The Series Solution*

Consider the power series expansions of the dynamics (4.1), and the running cost of index (4.2)

$$G_i(x,u) = \sum_{m=1}^{\infty} f_i^{(m)}(x) + \sum_{p=1}^{\infty} B_{ip}\, u^p + \sum_{m,p=1}^{\infty} f_{ip}^{(m)}(x)\, u^p, \quad i = 1, \ldots, n \tag{4.8}$$

$$r(x,u) = \sum_{m=2}^{\infty} r^{(m)}(x) + \sum_{p=2}^{\infty} R_p\, u^p + \sum_{m,p=1}^{\infty} r_p^{(m)}(x)\, u^p, \qquad i = 1, \ldots, n \tag{4.9}$$

where, (m) is the order of the functions in x, and p the power of scalar u. We also have that the constant $R_2 \in \mathbb{R}$ is nonzero: this is required for existence of stabilizing optimal control for the quadratic part of the system. Note that the order of the general running cost always starts at the quadratic order so that the function is always convex. Moreover, the running cost may have fewer terms, or the order of the terms may vary. For instance, for $m = p = 2$ and functions

$$\sum_{m=2}^{\infty} r^{(m)}(x) + \sum_{q=2}^{\infty} R_q\, u^q$$

the running cost is that of a linear quadratic regulator (LQR) problem.

Next, assume the solution form of the $V^*(x)$ and $u^*(x)$ functions as power series. We have

$$V(x) = V^{(2)}(x) + V^{(3)}(x) + \cdots + V^{(m)}(x) + \cdots \tag{4.10}$$

$$u(x) = u^{(1)} + u^{(2)} + \cdots + u^{(m-1)} + \cdots \tag{4.11}$$

Note that similarly here, the Lyapunov function $V(x)$ starts at quadratic order to retain positive definiteness. In addition, the terms of the value function (of any order) are the combination of all the possible monomials of that order. Notice also that order of control is always one lower than that of value function. This structure gives rise to the manner in which the value function and control are obtained. For example, the second order value function gives the linear control. The third order value function gives the second order control, the fourth order value function gives the third order control, and so on.

Recall the sufficient conditions of optimality and the system of equations (4.7). We would like to find the function $V(x)$ and $u(x)$ such that equations

$$0 = \sum_{i=1}^{n} G_i(x, u(x)) \frac{\partial V(x)}{\partial x_i} + r(x, u(x)) \tag{4.12}$$

$$0 = \sum_{i=1}^{n} \frac{G_i\big(x, u(x)\big)}{\partial u} \frac{\partial V(x)}{\partial x_i} + \frac{r\big(x, u(x)\big)}{\partial u} \tag{4.13}$$

are satisfied. Hence, let us substitute expansions (4.8), (4.9), (4.10), and (4.11) into equations (4.12), and (4.13). We shall group the resulting expansions of (4.12) and (4.13) based on their orders in x. The quadratic order of (4.12) is associated with the linear part of the system and gives the Riccati equation. Hence, the quadratic value function coefficient is obtained from the Riccati equation. Then, the first order of equation (4.13) gives the linear control $u^{(1)}(x)$ (which depends on the quadratic value function). For higher orders of value function, we shall consider the rest of the grouped terms of (4.12). Treating each order separately, we factor the grouped terms as linear combination of monomials of that order. Then the coefficients of the monomials are equated to zero and solved for the unknown coefficients of $V(x)$ as a system of linear equations.

In particular, we have that $V^{(m)}(x), m > 2$ are found from orders $3, \dots, m$ of equation (4.12). Controls $u^{(m-1)}(x), m > 3$ are then obtained from solving orders $2, \dots, m-1$ of equation (4.13). Moreover $u^{(m-1)}(x)$ when solved, are in terms of $V^{(m)}(x)$, hence the control at all orders can be found starting from the pair $V^{(2)}(x)$, and $u^{(1)}(x)$, if the solvability conditions are satisfied.

Let us now derive the value function and the control for any order m, and $m-1$ respectively [1]. We start by considering the linear part of the system (4.1), that is

$$\dot{x}_t = f_i^{(1)} + B_{i1}\, u^{(1)}(x), \qquad i = 1, \dots, n \tag{4.14}$$

Similarly, let us collect only the quadratic terms of the power series (4.9), such that

$$r^{(2)}(x,u) = r^{(2)}(x) + r_1^{(1)}(x)u^{(1)}(x) + R_2\left(u^{(1)}(x)\right)^2 \tag{4.15}$$

Hence, the cost functional of the linear system (4.14) becomes

$$J(u) = \int_0^\infty \left(r^{(2)}(x) + r_1^{(1)}(x)u^{(1)}(x) + R_2\left(u^{(1)}(x)\right)^2\right) dt \tag{4.16}$$

Equations (4.14) and (4.16) together are the familiar LQR problem. It is known that when B_{iq} and R_{qk} are nonzero (the pair $f_i^{(1)}$, B_{iq} are controllable), and $f_i^{(1)}$ is detectable, then the linear problem (4.14) is solvable [51], i.e. the Riccati solution gives the coefficient of $V^{(2)}(x)$ term. Further solving for linear control, the first order terms of equation (4.13) become

$$0 = \sum_{i=1}^{n} \frac{f_i^{(1)} + B_{i1}\, u^{(1)}(x)}{\partial u} \frac{\partial V(x)}{\partial x_i} + \frac{\left(r^{(2)}(x) + r_1^{(1)}(x)u^{(1)}(x) + R_2\left(u^{(1)}(x)\right)^2\right)}{\partial u}$$

$$\Rightarrow u^{(1)}(x) = -\frac{1}{2R_2}\sum_{i=1}^{n} B_{i1}\frac{\partial V^{(2)}(x)}{\partial x_i} - \frac{1}{2R_2} r_1^{(1)}(x) \tag{4.17}$$

Let us now approach the substitution and collection process of orders in a general sense. Suppose that we have already found the terms $V^{(3)}(x), \dots, V^{(m-1)}(x)$ and $u^{(2)}(x) + \cdots + u^{(m-2)}$, (Note that we have already obtained the pairs $V^{(2)}(x)$, and $u^{(1)}(x)$ from the linear part of the system). We are now interested in finding the next term of the solution series, $V^{(m)}(x)$, and $u^{(m-1)}(x)$. As described earlier, to solve for coefficients of $V^{(m)}(x)$, the grouped m^{th} order terms of (4.12) are needed. Consider all the m^{th} order terms of (4.12)

$$\sum_{i=1}^{n} f_i^{(1)}(x)\frac{\partial V^{(m)}(x)}{\partial x_i} + \sum_{i=1}^{n} B_{i1}\, u^{(m-1)}(x)\frac{\partial V^{(2)}(x)}{\partial x_i} + \sum_{i=1}^{n} B_{i1}\, u^{(1)}(x)\frac{\partial V^{(m)}(x)}{\partial x_i} + \tag{4.18}$$
$$R_2 u^{(1)}(x)u^{(m-1)}(x) + R_2 u^{(m-1)}(x)u^{(1)}(x) + u^{(m-1)}(x)\, r_1^{(1)}(x) = A^{(m)}(x)$$

where, $A^{(m)}(x)$ are all the m^{th} order terms with known coefficients. We also would like to solve for the coefficients of $u^{(m-1)}(x)$ from all the $(m-1)^{th}$ order terms of (4.13). Hence, also consider the $(m-1)^{th}$ terms of (4.13) given by

$$\sum_{i=1}^{n} B_{i1}\frac{\partial V^{(m)}(x)}{\partial x_i} + 2R_2 u^{(m-1)}(x) = B^{(m-1)}(x) \tag{4.19}$$

Similarly, $B^{(m-1)}(x)$ are the collection of $(m-1)^{th}$ order terms with known coefficients. Let us start by simplifying the equation (4.18). Factoring $\frac{\partial V^{(2)}(x)}{\partial x_i}$, and $\frac{\partial V^{(m)}(x)}{\partial x_i}$, and substituting for the linear control (4.17), we have

$$\sum_{i=1}^{n}\left(f_i^{(1)}(x) + B_{i1}\, u^{(1)}(x)\right)\frac{\partial V^{(m)}(x)}{\partial x_i} + \sum_{i=1}^{n} B_{i1}\, u^{(m-1)}(x)\frac{\partial V^{(2)}(x)}{\partial x_i} +$$
$$2R_2 u^{(m-1)}(x)\left(-\frac{1}{2R_2}\sum_{i=1}^{n} B_{i1}\frac{\partial V^{(2)}(x)}{\partial x_i} - \frac{1}{2R_2} r_1^{(1)}(x)\right) +$$

$$u^{(m-1)}(x)\, r_1^{(1)}(x) = A^{(m)}(x)$$

$$\Rightarrow \sum_{i=1}^{n} \left(f_i^{(1)}(x) + B_{i1}\, u^{(1)}(x)\right) \frac{\partial V^{(m)}(x)}{\partial x_i} + \sum_{i=1}^{n} B_{i1}\, u^{(m-1)}(x) \frac{\partial V^{(2)}(x)}{\partial x_i} +$$

$$\left(-\sum_{i=1}^{n} B_{i1} u^{(m-1)}(x) \frac{\partial V^{(2)}(x)}{\partial x_i} - u^{(m-1)}(x) r_1^{(1)}(x)\right) + u^{(m-1)}(x)\, r_1^{(1)}(x) = A^{(m)}(x)$$

Further, carrying out the cancellations, we obtain the simplified (4.18) as

$$\sum_{i=1}^{n} \left(f_i^{(1)}(x) + B_{i1}\, u^{(1)}(x)\right) \frac{\partial V^{(m)}(x)}{\partial x_i} = A^{(m)}(x) \tag{4.20}$$

Recognizing the linear part of the system $G_i^{(1)}(x,u) = f_i^{(1)}(x) + B_{i1}\, u^{(1)}(x)$, becomes

$$\sum_{i=1}^{n} G_i^{(1)}(x,u) \frac{\partial V^{(m)}(x)}{\partial x_i} = A^{(m)}(x)$$

$$\Rightarrow \frac{dV^{(m)}(x)}{dt} = A^{(m)}(x) \tag{4.21}$$

Given that $G_i^{(1)}(x,u)$ is asymptotically stable (i.e. it satisfies condition I, and $V(x)$ is a Lyapunov function), then by Lyapunov's theorem I (see Lyapunov [52] chapter 2, pages 71-79), there exists a unique solution $V^{(m)}(x)$ to equation (4.21). Let us now revisit these classical results.

Suppose that the linear dynamical system (4.14) can be written in the following form

$$\frac{dx}{dt} = M_{i1}x_1 + M_{i2}x_2 + \cdots + M_{in}x_n, \qquad i = 1, \ldots, n \tag{4.22}$$

where $M_{i1} \in \mathbb{R}$ are constant coefficients. Let us further define the following algebraic equation

$$\det \begin{vmatrix} (M_{11}-\lambda_1) & M_{12} & \cdots & M_{1n} \\ M_{21} & (M_{22}-\lambda_2) & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & (M_{nn}-\lambda_n) \end{vmatrix} = 0 \tag{4.23}$$

known as the *determinantal* equation, with unknowns $\lambda_i \quad i = 1, \dots, n$. It is important to mention that each λ_i corresponds to a solution of (4.22). Then, the following theorem gives the solvability condition of function $V(x)$ that we are interested in:

Theorem 4. [52] If the roots of the determinantal equation, λ_i $i = 1, \dots, n$ are such that, for a given positive integer $m = m_1 + m_2 + \cdots + m_n$, they cannot have a relation of the form

$$m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n = 0$$

with m_i coefficients being non-negative, then we will always be able to find a unique form $V(x)$ (of $(m)^{th}$ order in x), that satisfies the following equation

$$\sum_{i=1}^{n}(M_{i1}x_1 + M_{i2}x_2 + \cdots + M_{in}x_n)\frac{\partial V^{(m)}(x)}{\partial x_i} = A^{(m)}(x)$$

where $A(x)$ is any $(m)^{th}$ order known form.

This implies that any order m of the value function is obtainable. Moreover, for any order $V^{(m)}(x)$, control $u^{(m-1)}(x)$ can also be solved from equation (4.19). Reader may refer to Al'brekht [1] pages 1263-1265 for a similar derivation of the case when u is a vectorial quantity, though computation presented in the next chapter is for $u \in \mathbb{R}^m$.

As an example, consider the case where $m = 3$, and $n = 2$, where $x \in \mathbb{R}^2$. Then the assumed form of the value function becomes $V^{(3)}(x) = P_{30}\, x_1^3 + P_{21}\, x_1^2x_2 + P_{12}\, x_1x_2^2 + P_{03}\, x_2^3$, where the P values are the coefficients of different orders of $V^{(3)}(x)$. Suppose that equation (4.21) has a form

$$((M_{11} + M_{21} + M_{31} + M_{41})P_{30})\, x_1^3 + ((M_{12} + M_{22} + M_{32} + M_{42})P_{21})\, x_1^2x_2 +$$

$$((M_{13} + M_{23} + M_{33} + M_{43})P_{12})\, x_1x_2^2 + ((M_{14} + M_{24} + M_{34} + M_{44})P_{03})\, x_2^3 =$$

$$A_1x_1^3 + A_2x_1^2x_2 + A_3x_1x_2^2 + A_4x_2^3$$

with the constants M being the coefficients of the unknown optimal constants P (and the monomials), and A the known coefficients of the monomials. Then, the resultant equation can be expressed as a system of linear equations

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{bmatrix} P_{30} \\ P_{21} \\ P_{12} \\ P_{03} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

then the system can be solved for the unknown constants P with the nonsingular matrix M.

4.4 *A discussion on Convergence and Error*

Given that the solution to the HJB, the value function, is a power series, it is essential for the solution to be convergent. Specifically, if the infinite series is divergent, then any partial sum of the series is not an approximation of the optimal solution. Moreover, since the coefficients of the series are derived through the HJB equation, the form of the coefficients cannot be changed. In fact, we may only vary the coefficient values through choices of running cost weights, i.e. Q, R, and entries of the input matrix B. Note that the coefficients of the power series are functions of Q, R, and B. Hence, we may conclude that the radius of convergence of such series is formed through choices of such constants.

Conversely, if the infinite series is convergent, then the partial sums of the solution can be considered an approximation of the optimal solution. Though, this approximation is valid only within the radius of convergence of the infinite series. Specifically, equation $\frac{dv}{dt} = -r(x,u)$ is satisfied only within the radius of convergence of the infinite power series. Error may then arise when the series is truncated. i.e. the error will vary in different regions of the state-space. In fact, the error due to truncation of higher orders of the solution series decreases as we approach the origin. In general, a finite truncated series that satisfies the HJB equation approximately tells us that the HJB equation has a solution in form of a power series within a radius of convergence. Another numerical issue is the region of attraction of the origin given by the Lyapunov equation

$$\sum_{i=1}^{n} G_i(x_t, u(x_t)) \frac{\partial V(x_t)}{\partial x_i} < 0$$

where $V(x_t) > 0$. It is possible that increasing the order of the approximation causes the region where the Lyapunov equation is valid to shrink [53]. This is both undesirable and counterintuitive because, when increasing the order of an approximation for solution accuracy, the terms cancellation in some regions may actually cause the region of attraction to shrink.

CHAPTER 5
Spacecraft Attitude Control and the Problem of Thrust Uncertainty

5.1 *Stochastic Satellite Attitude Stabilization and Control*

In this chapter, we provide a method of active recovery and correction for spacecraft attitude thrusters with thrust uncertainty. The method is a component of an active fault detection, isolation, and recovery (FDIR) strategy for the attitude determination and control system (ADCS). The provided optimal attitude stabilization method can be extended for tracking and control applications, as well as, considering the attitude kinematics subsystem. Here we are concerned with stabilization of rotational rates of the spacecraft while achieving desired criteria (i.e. minimum fuel consumption) under thrust uncertainty. In fact, by accounting for the generated thrust uncertainty, we reduce the error in system's state, as well as achieving lower optimality error. This is specifically desirable in detumbling maneuvers, spacecraft proximity operations, as well as low thrust spacecraft maneuvers. Other applications include, stabilization of spacecraft attitude during rendezvous, or stabilization during in-orbit servicing operations. Thrust fluctuations and deviation from the commanded mean torque can result in undesirable effects such as excessive fuel consumption (limiting the lifetime of the mission), error in precision pointing of satellite antenna, or collisions in extreme cases. Therefore, generation of precise torques and compensating for uncertainty through design of stochastic controllers is desired both for safety, as well as optimization of mission parameters. We further demonstrate, through numerical experiments and simulations, that the stochastic controllers will have a lower optimality error on average. We have shown that for systems with assumption of no uncertainty, linear controllers can be made optimal. However, in presence of uncertainty and noise, these controllers are no longer optimal. Hence, nonlinear stochastic controllers are required to achieve the minimum cost.

Let us define the expected cost functional for the optimal attitude stabilization problem as

$$J(u) = \mathbb{E}\left[\int_{0}^{\infty} r(x_t, u(t))\, dt \mid x_{t=0} = x_0\right] \tag{5.1}$$

where $\mathbb{E}$ is an expected value with respect to probability measure $\mathbb{P}$, $\omega = x \in \mathbb{R}^{3\times1}$ and $u \in \mathbb{R}^{m\times1}$ are the angular velocity and control input respectively, and $r(x, u)$ is the running cost. Specifically, m is the number of thruster pairs of the model. The cost functional (5.1) is the expectation over all the trajectories starting at the initial state x_0. We would like to find a control trajectory u, in an infinite horizon setting, such that it minimizes the cost functional. The cost functional (5.1) quantifies the total scaled energy of the angular velocity x, and the control input u. The quadratic running cost is then given by $r(x, u) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u$, where $R > 0$, $R \in \mathbb{R}^{m\times m}$, and $Q \geq 0$, $Q \in \mathbb{R}^{3\times3}$ are the constant matrices penalizing the input and the state respectively.

Recall the spacecraft attitude model with multiplicative noise (see section 2.3). The idea behind the model is to let the uncertainty propagation by a thruster be modeled by Gaussian white noise process. The system equation is given by

$$dx_t = G(x_t, \bar{u}(t))\, dt + \sigma(\bar{u}(t)) dW_t \tag{5.2}$$

$$\sigma(\bar{u}) = \varepsilon B \begin{bmatrix} \bar{u}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{u}_m \end{bmatrix} \tag{5.3}$$

where $G(x, u) = f(x) + Bu$, and $f(x)$, given by (5.4), is the drift vector field describing the rotational dynamics, $\bar{u} \in \mathbb{R}^{m\times1}$ is the nominal (commanded) control,

$$f(x) = \begin{bmatrix} \dfrac{I_{22} - I_{33}}{I_{11}}\, x_2 x_3 \\ \dfrac{I_{33} - I_{11}}{I_{22}}\, x_3 x_1 \\ \dfrac{I_{11} - I_{22}}{I_{33}}\, x_1 x_2 \end{bmatrix} \tag{5.4}$$

and $B = I^{-1}b$, $B \in \mathbb{R}^{3\times m}$, where b is a matrix with its columns being the axes in which the corresponding torques are applied about (see section 2.2). The constant $\varepsilon \geq 0, \varepsilon \in \mathbb{R}$ is a parameter scaling the thruster uncertainty effects. Here we assume that I_{11}, I_{22}, I_{33}, are the diagonal components of $I \in \mathbb{R}^{3\times 3}$, the principal moments of inertia matrix for the spacecraft. We also have that $W_t, t \geq 0$ is the m-dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\sigma(u)$ denotes the diffusion coefficient. In particular $\sigma(\bar{u}(t))dW_t$ term models the thrust error. In fact, W_t is m-dimensional because uncertainty is unique to each thruster. Moreover, we have that a generated torque is due to net force produced by a thruster pair as shown in section 2.2. This means that the torque uncertainty is due to uncertainty from two thrusters.

To find the optimal stabilizing control that minimizes (5.1) we formulate a HJB equation associated with the nonlinear SDE (5.2). We will then use the Al'brekht method [1] to find the value function solution to the HJB PDE, as well as providing a stochastic optimal control which is close to optimal around the origin. We formulate the stationary HJB equation (5.5), along with the infinitesimal generator of the diffusion (5.6) defined by the SDE (5.2). Note that $a_{i,j}(u) = (\sigma(u)\sigma(u)^T)_{i,j}$ is the covariance matrix, $a(u) \in \mathbb{R}^{3\times 3}$, and σ is defined by (5.3). The superscript u denotes the dependency of the generator on control. From now on, we shall refrain from using the nominal control's overline notation, and reserve to simply writing u.

$$\min_u \{\mathcal{L}^u V(x) + r(x,u)\} = 0 \tag{5.5}$$

$$\mathcal{L}^u V(x) = \sum_{i=1}^{n} G_i(x,u)\frac{\partial V(x)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(u)\frac{\partial^2 V(x)}{\partial x_i \partial x_j} \tag{5.6}$$

Applying the generator (5.6) to the value function, the HJB equation is written as

$$\min_u \left\{ \sum_{i=1}^{n} G_i(x,u)\frac{\partial V(x)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(u)\frac{\partial^2 V(x)}{\partial x_i \partial x_j} + r(x,u) \right\} = 0 \tag{5.7}$$

where $n = 3$ for dynamics (5.2). Further substituting for the dynamics vector $G(x,u)$, and separating the terms that are independent of the control u, we rewrite HJB (5.7) as

$$0 = f(x)^T \frac{\partial V(x)}{\partial x} + \frac{1}{2} x^T Q x + \min_u \left\{ (Bu)^T \frac{\partial V(x)}{\partial x} + \frac{1}{2} trace\left(\sigma\sigma^T \frac{\partial^2 V(x)}{\partial x^2} \right) + \frac{1}{2} u^T R u \right\} \tag{5.8}$$

Looking closer at the formulated HJB PDE above, the $\frac{1}{2} trace\left(\sigma\sigma^T \frac{\partial^2 V(x)}{\partial x^2}\right)$ term carrying the noise effects is the difference between a HJB for deterministic attitude dynamics and (5.2). Instead of using lengthy notation, we will reserve to expressing the HJB "noise terms" using a briefer notation. Let $H: \mathbb{R} \to \mathbb{R}^{m \times m}$ be a diagonal second order differential function defined as

$$H[.] = \varepsilon^2 DiagonalMatrix(\, Diagonal(\, B^T Hessian[.]\, B\,)\,) \tag{5.9}$$

Then, expanding $trace\left(\sigma\sigma^T \frac{\partial^2 V(x)}{\partial x^2}\right)$ implies that

$$u^T H[V(x)] u = trace\left(\sigma\sigma^T \frac{\partial^2 V(x)}{\partial x^2} \right)$$

To approximate the solution of the HJB (5.8), we assume that the HJB satisfies a solution in form of convergent power series. Hence, we construct a power series representation of the value function $V(x)$, and the optimal control $u(x)$ [1]. Then, Al'brekht method tells us that the optimal control $u^*(x)$ and optimal cost function $V^*(x)$ infinite series satisfy the following HJB analog equations within their radius of convergence such that

$$0 = f(x)^T \frac{\partial V^*(x)}{\partial x} + \frac{1}{2} x^T Q x + (Bu^*)^T \frac{\partial V^*(x)}{\partial x} + \frac{1}{2} u^{*T} H\big(V^*(x)\big) u^* + \frac{1}{2} u^{*T} R u^* \tag{5.10}$$

$$0 = B^T \frac{\partial V^*(x)}{\partial x} + H\big(V^*(x)\big) u^* + R u^* \tag{5.11}$$

Considering equations (5.10), and (5.11), we wish to find partial sums of the infinite series $V^*(x)$, $u^*(x)$ that satisfy the expanded HJB. To find such partial sums, we expand the rest of the HJB equation, that is the dynamics and the running cost. Then the HJB analog (5.8) becomes a collection of terms from the truncated series in different orders of x. Hence, if equations (5.10) and (5.11) can be solved for the value function $V(x)$ and the control $u(x)$ at different orders, then the expansions $V(x)$ and $u(x)$ satisfy the HJB locally around the origin. Note that the series truncation error (within the radius of convergence of the infinite series) is negligible close to the origin, hence, $V(x)$ and $u(x)$ satisfy the HJB locally. We will demonstrate how the partial sums $u(x)$, and $V(x)$ are obtained considering the spacecraft dynamics.

Let us begin by assuming the solution form of $u(x)$, and $V(x)$. Note that any expansion of the value function and the running cost must retain their positive definite property. Thus, the lowest order of any expansion of the value function or the running cost will pose a quadratic form. Consider the expansion of the value function and the optimal control

$$V(x) = \frac{1}{2}x^T Px + V^{(3)}(x) + V^{(4)}(x) + \cdots \tag{5.12}$$

$$u(x) = Kx + k^{(2)}(x) + k^{(3)}(x) + \cdots \tag{5.13}$$

where $V^{(m)}(x)$ is a homogenous polynomial of $(m)^{th}$ order in x, $\frac{1}{2}x^T Px = V^{(2)}(x)$ is the second order value function, P is a positive definite symmetric matrix, $k^{(m-1)}(x)$ is the $(m-1)^{\text{th}}$ order nonlinear optimal control term, and $Kx = k^{(1)}(x)$ is the linear optimal feedback. The value function may be expanded up to the $(m)^{th}$ order, whereas the corresponding control is always truncated at the $(m-1)^{th}$ order (see chapter 4).

It is important to note that the quadratic order of (5.12) and the linear order of (2.13) give the stabilizing control to the linear part of the system. In other words, if the linear part of the system is controllable, under the additional conditions (given in consideration of the stochastic system) in section 5.2, we may then obtain the rest of the higher orders of $u(x)$ and $V(x)$ successively as in Al'brekht [1]. We will provide the corresponding higher order stochastic solvability conditions for the third and fourth order value function in section 5.3.

Next, let us expand the dynamics, and the running cost as power series around the origin

$$G\big(x,u(x)\big) = Fx + f^{(2)}(x) + f^{(3)}(x) + \cdots + BKx + Bk^{(2)}(x) + Bk^{(3)}(x) + \cdots \tag{5.14}$$

$$r\big(x,u(x)\big) = \frac{1}{2}x^T Qx + \frac{1}{2}(Kx)^T RKx + r^{(3)}(x) + r^{(4)}(x) + \cdots \tag{5.15}$$

Note that the spacecraft dynamics are given by (5.4) where the drift term of (5.2) $f(x) = f^{(2)}(x)$ is of second order, and the linear part $Fx = 0$. We shall substitute series (5.12), (5.13), (5.14), and (5.15) into the system of equations (5.10), (5.11). Collecting and grouping the different orders of equation (5.10), we have

$$\begin{aligned} &f^{(2)}(x)^{\mathrm{T}}\ \nabla\big[V^{(2)}(x) + \cdots + V^{(m)}(x)\big] \\ &+\ B\left(Kx + k^{(m)}(x) + \cdots + k^{(m-1)}(x)\right)^T\ \nabla\big[V^{(2)}(x) + \cdots + V^{(m)}(x)\big] \\ &+ \frac{1}{2}\left(Kx + k^{(2)}(x) + \cdots + k^{(m-1)}(x)\right)^T H\big[V^{(2)}(x) + \cdots \\ &+ V^{(m)}(x)\big]\left(Kx + k^{(2)}(x) + \cdots + k^{(m-1)}(x)\right) \\ &+ \left(\frac{1}{2}x^T Qx + \frac{1}{2}(Kx)^T RKx + r^{(3)}(x) + \cdots + r^{((m-1)^2)}(x)\right) \\ &+ O|x|^{(2m-1)} = 0 \end{aligned} \tag{5.16}$$

In fact, for equation (5.16) to hold, we have assumed that $V(x)$, and $u(x)$ are the optimal cost, and the optimal control respectively. In other words, solving for a truncated $V(x)$ from equation (5.16), is indeed solving for an approximation of the optimal value function. We have that since the m^{th} partial sum, $V^{(m)}(x)$, satisfies the HJB as $m \to \infty$, then within the radius of convergence of the infinite series, the truncated value function satisfies the equation (5.16) locally around the origin.

From (5.11), we also have

$$\begin{aligned} &B^T\nabla\big[V^{(2)}(x) + \cdots + V^{(m)}(x)\big] \\ &+ H\big[V^{(2)}(x) + \cdots + V^{(m)}(x)\big]\left(Kx + k^{(2)}(x) + \cdots + k^{(m-1)}(x)\right) \\ &+ R\left(Kx + k^{(2)}(x) + \cdots + k^{(m-1)}(x)\right) + O|x|^m = 0 \end{aligned} \tag{5.17}$$

Similarly, for equation (5.17) to hold, we have made the assumption that $u(x)$, and $V(x)$ are optimal. Hence, solving for $(m-1)^{th}$ partial sum of the convergent control series from (5.17), is solving for an approximation of the optimal control that satisfies the HJB (5.8) as $m \to \infty$. The obtained truncated control is valid and stabilizing within the radius of convergence of the infinite series, and within the region of attraction given by the Lyapunov equation (see section 5.4). Hence, the truncated $u(x)$ series can be considered an approximation to the stabilizing nonlinear optimal control in a neighborhood of the origin.

5.2 *Linear Stochastic Control*

In this section, we derive the linear stochastic control that stabilizes the dynamics. Arranging the terms of (5.16) based on their degree in x, we separate each order of m and solve the optimal control expressions in terms of the value function expansion. For instance, the linear stochastic optimal feedback term is obtainable from the linear terms of (5.16). This term is dependent on the second order value function coefficient P. The linear stochastic optimal feedback gain K is

$$K = -\left(H\left(V^{(2)}(x)\right) + R\right)^{-1} (PB)^T \tag{5.18}$$

Since the linear control (5.18) is dependent on the second order value function, entries of the matrix P are needed. Expanding and organizing (5.16), by substitution of (5.18) for the grouped quadratic terms, we arrive at equation (5.19). We may now point out that the second order terms have formed the following algebraic Riccati equation (ARE)

$$Q - (PB)\left(H\left(V^{(2)}(x)\right) + R\right)^{-1} (PB)^T = 0 \tag{5.19}$$

The solution to the ARE (5.19) is the symmetric positive definite matrix P, where Q, and R are the matrices defined in (5.1). The ARE (5.19) differs from its counterpart, ARE for a deterministic model, via the $H\left(V^{(2)}(x)\right)$ term arising from diffusion terms.

The Riccati equation is a well-studied area of research and its solvability and existence conditions are well known. Consider the following ARE associated with a general deterministic linear system of the form $\dot{x} = Ax + Bu$,

$$PA + A^T P - PBR^{-1}(PB)^T + Q = 0 \tag{5.20}$$

such that $A \epsilon \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$. We refer the readers to the work of V. Kucera [51] where proofs of existence and uniqueness of solution to the ARE are given. A Hermitian solution P to the ARE (5.20) exists if for $Q = C^T C$, the pair (A, C) is detectable, and the pair (A, B) is controllable. For equation (5.19), the conditions can be inferred from the results of Theorem 2.1 of Wonham [54]. The form of the ARE studied by Wonham is

$$PA + A^T P - PBR^{-1}(PB)^T + \Pi(P) + Q = 0 \tag{5.21}$$

where $\Pi(P)$ is a linear map from the space of symmetric $n \times n$ matrices onto itself. The linear map $\Pi(P)$ is such that if P is positive semi-definite, then so is $\Pi(P)$. The positive definite solution P to (5.21) exists if (similar to (5.20)) for $Q = C^T C$, the pair (A, C) is detectable, and the pair (A, B) is controllable, with the additional condition that

$$\inf_K \left| \int_0^\infty e^{(A+BK)^T t} \Pi(1_{n\times n}) e^{(A+BK)t}\, dt \right| < 1. \tag{5.22}$$

For the ARE (5.19) associated with the linear stochastic dynamics (5.2), we have that $A \equiv 0$ and $\Pi(P) = PBR^{-1}\left(R^{-1} + H(B^T PB)\right)^{-1} R^{-1} B^T P$ through the Woodbury identity. When B and R are diagonal and $m = 3 = \dim(x)$ for the optimization problem (5.1), (5.2), the condition (5.22) for solvability of the ARE (5.19) becomes, for $i = 1, \dots, m$,

$$\left| \int_0^\infty \frac{B_i^2/R_i}{1 + B_i^2/R_i}\, e^{2\lambda_{max}^K t}\, dt \right| < 1 \implies \frac{B_i^2/R_i}{1 + B_i^2/R_i} \frac{1}{2|\lambda_{max}^K|} < 1 \tag{5.23}$$

where B_i, R_i are the diagonal entries of B, R, respectively, and λ_{max}^K is the maximum eigenvalue of $(A + BK)$ for an aggressive linear control gain K such that the eigenvalues of $(A + BK)$ are all negative. Such a control exists by the stabilizability assumption. Although AREs are usually hard to solve algebraically, they are certainly obtainable depending on the assumed form of the matrices. Note that a numerical tool to approximate the solution of the ARE is the linear matrix inequalities (LMI) method. Rami and Zhou [55] have highlighted the LMI method for the specific case of (5.11) with a scalar Brownian motion. Solvability conditions have been provided in the same work.

5.3 *Nonlinear Stochastic Control*

Let us now derive the higher order control terms. Similarly, we separate and group the terms of order $m \geq 2$ in equation (5.17). Separating and arranging the quadratic and cubic terms of (5.17) yield expressions containing $k^{(2)}(x)$, and $k^{(3)}(x)$. Solving these quadratic and cubic expressions, the control terms are obtained as

$$k^{(2)}(x) = -\left(H\left(V^{(2)}(x)\right) + R\right)^{-1}\left((B^T \nabla V^{(3)}(x) + H\left(V^{(3)}(x)\right)Kx\right) \tag{5.24}$$

$$\begin{aligned} k^{(3)}(x) = -\left(H\left(V^{(2)}(x)\right) + R\right)^{-1} (B^T \nabla V^{(4)}(x) + H\left(V^{(3)}(x)\right)\left(k^{(2)}(x)\right) \\ + H\left(V^{(4)}(x)\right)Kx) \end{aligned} \tag{5.25}$$

Clearly, both (5.24) and (5.25) depend on the value function terms $V^{(3)}(x)$ and $V^{(4)}(x)$. Hence, we will solve for the coefficients of the homogenous polynomials

$$V^{(3)}(x) = \sum_{i+j+k=3} p_{ijk}\, x_1^i x_2^j x_3^k$$

$$V^{(4)}(x) = \sum_{i+j+k=4} p_{ijk}\, x_1^i x_2^j x_3^k$$

where p_{ijk} are the optimal coefficients of the monomials. Note that obtaining the 3rd order and 4th order value function polynomials requires a more careful consideration. Here we assume that we have already obtained the solution to the ARE formed by the quadratic terms of (5.16), hence, the linear control equation is known. Manipulating the linear control coefficient, we have

$$P = -B^T\left(H\left(V^{(2)}(x)\right) + R\right)K \tag{5.26}$$

Substituting (5.26) into the cubic and quartic terms of (5.16) and further simplifying, we obtain

$$0 = f^{(2)}(x)^T\nabla V^{(2)}(x) + \left(B(Kx)\right)^T\nabla V^{(3)}(x) + \frac{1}{2}(Kx)^T H\left(V^{(3)}(x)\right)(Kx) \tag{5.27}$$

$$0 = f^{(2)}(x)^T\nabla V^{(3)}(x) + \left(B(Kx)\right)^T\nabla V^{(4)}(x) + \frac{1}{2}(Kx)^T H\left(V^{(4)}(x)\right)(Kx) \tag{5.28}$$

The only unknowns in equations (5.27) and (5.28) are the value function terms $V^{(3)}$ and $V^{(4)}$ respectively. Let us now define the linear operators for (5.27) and (5.28)

$$\tilde{\mathcal{L}}V^{(3)}(x) := (BKx)^{\mathrm{T}}\nabla V^{(3)}(x) + \frac{1}{2}(Kx)^T H\left(V^{(3)}(x)\right)(Kx) \tag{5.29}$$

$$\tilde{\mathcal{L}}V^{(4)}(x) \coloneqq (BKx)^{\mathrm{T}}\nabla V^{(4)}(x) + \frac{1}{2}(Kx)^T H\left(V^{(4)}(x)\right)(Kx) \tag{5.30}$$

To examine the solvability of (5.27) and (5.28) for $V^{(3)}(x)$ and $V^{(4)}(x)$ polynomials, we examine the eigenvalues of the corresponding operators (5.29) and (5.30). The notion of linear operator, or the homological equation was introduced by Arnold [56] following the work of Poincare in normal forms theory. Equations (5.27) and (5.28) are solvable for $V^{(3)}(x)$ and $V^{(4)}(x)$ if the eigenvalues of the corresponding operators are nonzero. In equation (5.2) for $\varepsilon = 0$, the system becomes deterministic. As a result, linear operators (5.29) and (5.30) only contain first order differential operators. For the linear system with matrix BK, the eigenvalues of the corresponding linear operators (5.29) and (5.30) are

$$\lambda_i + \lambda_j + \lambda_k \tag{5.31}$$

$$\lambda_i + \lambda_j + \lambda_k + \lambda_l \tag{5.32}$$

respectively, where $\lambda_i, \lambda_j, \lambda_k, \lambda_l, i, j, k, l \in \{1, \dots, n\}$, are the eigenvalues of BK [44]. The condition for solvability is such that these sums of eigenvalues of BK are nonzero. This is also the non-resonance condition for a dynamical system [56], [57]. In the case of the stochastic system, the solvability condition is complicated due to the second order operator. We may now state the following lemma regarding the solvability conditions of (5.27), and (5.28).

Lemma 1. Suppose that the linear gain BK in (5.2) is diagonal with distinct eigenvalues, then for diagonal Q, and R of (5.1), the eigenvalues of the linear operators (5.29) and (5.30) are

$$\lambda_i + \lambda_j + \lambda_k + \varepsilon^2\left(\delta_{ij}\lambda_i\lambda_j + \delta_{ik}\lambda_i\lambda_k + \delta_{kj}\lambda_k\lambda_j\right) \tag{5.33}$$

$$\lambda_i + \lambda_j + \lambda_k + \lambda_l + \varepsilon^2\left(\delta_{ij}\lambda_i\lambda_j + \delta_{ik}\lambda_i\lambda_k + \delta_{il}\lambda_i\lambda_l + \delta_{jk}\lambda_j\lambda_k + \delta_{jl}\lambda_j\lambda_l + \delta_{kl}\lambda_k\lambda_l\right) \tag{5.34}$$

respectively, where δ_{ij} is the Kronecker delta. Then (5.27) and (5.28) are solvable if these eigenvalues of the operators are nonzero.

Proof Consider the HJB equation (5.8) associated with the state equation (5.2). Let P^1 be given by the ARE resulting from the second order polynomial in x of (5.8), and K^1 is the corresponding optimal linear gain. Using P^1, $V^{(3)}$ is obtained by solving the 3rd order polynomial in x of (5.8)

$$-f(x)^T P^1 x = \left(Bu^1(x)\right)^T \nabla_x V^{(3)}(x) + \frac{\varepsilon^2}{2} trace\left[BU^1(x)\left(BU^1(x)\right)^T \nabla_x^2 V^{(3)}(x)\right] \tag{5.35}$$

where $u^1(x) \coloneqq K^1 x$, K^1 is the linear gain (5.18), and U^1 is the corresponding diagonal $n \times n$ matrix constructed using u^1. For a twice differentiable function $\varphi: R^n \to R$, we define the differential operator

$$\tilde{\mathcal{L}}^2\varphi(x) \coloneqq \left(Bu^1(x)\right)^T \nabla_x\varphi(x) + \frac{\varepsilon^2}{2} trace\left[BU^1(x)\left(BU^1(x)\right)^T \nabla_x^2\varphi(x)\right] \tag{5.36}$$

The second order partial differential equation (5.35) has a solution $V^{(3)}$ if the operator $\tilde{\mathcal{L}}^2$ defined by (30) has nonzero eigenvalues. We want to determine the eigenvalues of $\tilde{\mathcal{L}}^2$.

Let (w^a, λ^a), $a = 1, \dots, n$, denote a left eigenvector of the $n \times n$ matrix BK^1 and corresponding eigenvalue ($n = 3$ in this chapter). If the eigenvalues of BK^1 are real-valued, then the left and right eigenvalues are equal. The cubic polynomial in x, $V^{(3)}(x)$ can then be represented by

$$V^{(3)}(x) = \langle \alpha, x\rangle\langle \beta, x\rangle\langle \gamma, x\rangle,$$

where $\alpha, \beta, \gamma \in \mathbb{R}^n$. We can also represent $V^{(3)}(x)$ using a basis constructed using the left eigenvectors of BK^1 as

$$V^{(3)}(x) = \sum_{i,j,k=1}^{n} c_{ijk}^{(3)} V_{ijk}^{(3)}(x) \tag{5.37}$$

where $V_{ijk}^{(3)}(x) = \langle w^i, x\rangle\langle w^j, x\rangle\langle w^k, x\rangle$. If we can show that $\tilde{\mathcal{L}}^2 V_{ijk}^{(3)}(x) = \nu^{ijk} V_{ijk}^{(3)}(x)$ for some ν^{ijk}, then we can conclude that ν^{ijk} is the ijk^{th} eigenvalue of L^2. Consider the first order operator in (2.36) acting on a basis function of $V^{(3)}(x)$. For brevity, we will denote the matrix BK^1 by $\hat{B}$.

$$\begin{aligned}
&\left(\hat{B}x\right)^T \nabla_x V_{ijk}^{(3)}(x) \\
&= x^T \hat{B}^T \nabla_x \left[\langle w^i, x\rangle\langle w^j, x\rangle\langle w^k, x\rangle\right] \\
&= x^T \hat{B}^T \left[\left(w^i\right)^T \langle w^j, x\rangle\langle w^k, x\rangle + \left(w^j\right)^T \langle w^i, x\rangle\langle w^k, x\rangle + (w^k)^T \langle w^i, x\rangle\langle w^j, x\rangle\right] \\
&= x^T \left[\lambda^i \left(w^i\right)^T \langle w^j, x\rangle\langle w^k, x\rangle + \lambda^j \left(w^j\right)^T \langle w^i, x\rangle\langle w^k, x\rangle + \lambda^k (w^k)^T \langle w^i, x\rangle\langle w^j, x\rangle\right] \\
&= \left[\lambda^i \langle w^i, x\rangle\langle w^j, x\rangle\langle w^k, x\rangle + \lambda^j \langle w^i, x\rangle\langle w^j, x\rangle\langle w^k, x\rangle + \lambda^k \langle w^i, x\rangle\langle w^j, x\rangle\langle w^k, x\rangle\right] \\
&= \left(\lambda^i + \lambda^j + \lambda^k\right) V_{ijk}^{(3)}(x)
\end{aligned} \tag{5.38}$$

where, w^ls, $l = i, j, k$ are the left eigenvectors of $\hat{B}$. If the state is deterministic, then (5.35) is just a first order partial differential equation and the condition for solvability of (5.35) will be $\left(\lambda^i + \lambda^j + \lambda^k\right)$ is nonzero for all $i, j, k = 1, \dots, n$, where $\lambda^i, \lambda^j, \lambda^k$ are the eigenvalues of $\hat{B} = BK^1$. This result is the same as that in [44].

Next, consider the second order operator in (5.36) acting on a basis function of $V^{(3)}(x)$. By the hypothesis, we have that matrices B, Q and R are diagonal. From (5.18), K^1 is also diagonal, and hence so is BK^1. If BK^1 is diagonal, then its eigenvectors are $w^i = e^i$, $i = 1, \dots, n$, where e^i is a unit vector in the i-direction. Therefore, $w^a = w_a^i \delta_{ip}$ for $i = 1, \dots, n$, where δ_{ia} is the Kronecker delta, which equals 1 when $i = a$, zero otherwise.

Let $\tilde{B}(x) := BU^1(x)$, which is a diagonal $n \times n$ matrix. Recall that $U^1(x)$ is a diagonal $n \times n$ matrix with diagonal entries being the vector $K^1 x$. The second order differential operator (5.36) acting on a basis function $V_{ijk}^{(3)}(x)$ is

$$trace\left[BU^1(x)\left(BU^1(x)\right)^T \nabla_x^2 V_{ijk}^3(x)\right] = trace\left[\tilde{B}(x)\tilde{B}(x)^T \nabla_x^2 V_{ijk}^3(x)\right]$$

$$= \tilde{B}_{pq}(x)\, \partial_{pr}^2 V_{ijk}^3(x)\, \tilde{B}_{rq}(x)\, \delta_{pq}\delta_{rq} = B_{pa}K_{ab}x_b\, \partial_{rr}^2 V_{ijk}^3(x)\, B_{pc}K_{cd}x_d\, \delta_{pr}$$

$$= x_b K_{ab} B_{pa}\left[2w_r^i w_r^j\, \delta_{ir}\delta_{jr}\, \langle w^k, x\rangle + 2w_r^i w_r^k\, \delta_{ir}\delta_{kr}\, \langle w^j, x\rangle \right.$$
$$\left. + 2w_r^j w_r^k\, \delta_{jr}\delta_{kr}\, \langle w^i, x\rangle\right] B_{pc}K_{cd}x_d\, \delta_{pr}$$

$$= x_b K_{ab} B_{pa}\left[2w_p^i w_r^j\, \delta_{ip}\delta_{jr}\, \langle w^k, x\rangle + 2w_p^i w_r^k\, \delta_{ip}\delta_{kr}\, \langle w^j, x\rangle \right.$$
$$\left. + 2w_p^j w_r^k\, \delta_{jp}\delta_{kr}\, \langle w^i, x\rangle\right] B_{rc}K_{cd}x_d\, \delta_{pr} \tag{5.39}$$

$$= x_b K_{ab} B_{pa}\left[2w_p^i \lambda^i\, \lambda^j w_r^j\, \delta_{ip}\delta_{jr}\, \langle w^k, x\rangle \right.$$
$$\left. +2w_p^i \lambda^i\, \lambda^k w_r^k\, \delta_{ip}\delta_{kr}\, \langle w^j, x\rangle + 2w_p^j \lambda^j\, \lambda^k w_r^k\, \delta_{jp}\delta_{kr}\, \langle w^i, x\rangle\right] B_{rc}K_{cd}x_d\, \delta_{pr}$$

$$= 2\left(\lambda^i\lambda^j\delta_{ij} + \lambda^i\lambda^k\delta_{ik} + \lambda^j\lambda^k\delta_{jk}\right)\langle w^i, x\rangle\langle w^j, x\rangle\langle w^k, x\rangle$$

$$= 2\left(\lambda^i\lambda^j\delta_{ij} + \lambda^i\lambda^k\delta_{ik} + \lambda^j\lambda^k\delta_{jk}\right)V_{ijk}^{(3)}(x),$$

by collapsing the Kronecker delta. Collecting (5.36), (5.38) and (5.39), we have

$$\left(Bu^1(x)\right)^T \nabla_x V_{ijk}^3(x) + \frac{\varepsilon^2}{2} trace\left[BU^1(x)\left(BU^1(x)\right)^T \nabla_x^2 V_{ijk}^3(x)\right]$$
$$= \left(\lambda^i + \lambda^j + \lambda^k\right)V_{ijk}^{(3)}(x) + \varepsilon^2\left(\lambda^i\lambda^j\delta_{ij} + \lambda^i\lambda^k\delta_{ik} + \lambda^j\lambda^k\delta_{jk}\right)V_{ijk}^{(3)}(x).$$

Hence, for B, Q, and R diagonal, the second order operator (5.36) has eigenvalues

$$\left(\lambda^i + \lambda^j + \lambda^k\right) + \varepsilon^2\left(\lambda^i\lambda^j\delta_{ij} + \lambda^i\lambda^k\delta_{ik} + \lambda^j\lambda^k\delta_{jk}\right),$$

$i, j, k \in \{1, \dots, n\}$, where λ^ls are the eigenvalues of BK^1 as desired. Solution to (5.35) exists when these eigenvalues are nonzero.

Next, consider the fourth order polynomial in x of (5.16)

$$-f(x)^T\nabla_x V^{(3)} = \left(Bu^1(x)\right)^T\nabla_x V^{(4)}(x) + \frac{\varepsilon^2}{2} trace\left[BU^1(x)\left(BU^1(x)\right)^T\nabla_x^2 V^{(4)}(x)\right] \quad (5.40)$$

Equation (5.40) has a solution $V^{(4)}$ if the second order differential operator on the RHS of (5.40) has nonzero eigenvalues. We can similarly express $V^{(4)}(x)$ in terms of basis functions constructed using the left eigenvectors of BK^1

$$V^{(4)}(x) = \sum_{i,j,k,l=1}^{n} c_{ijkl}^{(3)} V_{ijkl}^{(3)}(x) \quad (5.41)$$

where $V_{ijkl}^{(4)}(x) = \langle w^i, x\rangle\langle w^j, x\rangle\langle w^k, x\rangle\langle w^l, x\rangle$. By the same analysis as for $V^{(3)}(x)$, using the basis function of $V^{(4)}(x)$, the eigenvalues of the differential operator on the RHS of (5.40) become

$$\left(\lambda^i + \lambda^j + \lambda^k + \lambda^l\right) + \varepsilon^2\left(\lambda^i\lambda^j\delta_{ij} + \lambda^i\lambda^k\delta_{ik} + \lambda^i\lambda^l\delta_{il} + \lambda^j\lambda^k\delta_{jk} + \lambda^j\lambda^l\delta_{jl} + \lambda^k\lambda^l\delta_{kl}\right)$$

$i, j, k, l \in \{1, \dots, n\}$ as desired. If all eigenvalues are nonzero, then the linear operator is invertible, hence the corresponding PDE can be solved given any forcing function. ■

Equations (5.27) and (5.28) yield the value function $V^{(3)}(x)$ and $V^{(4)}(x)$, where these orders of value function are substituted in (5.24) and (5.25) to solve for controls $k^{(2)}(x)$, and $k^{(3)}(x)$. In this manner, we find the higher order controls. In general, consider the pairs $V^{(m)}(x)$, and $u^{(m-1)}(x)$, $\forall\, m$. Suppose $m = 2$, and that the linear system

$$dx_t = B\bar{u}(t)\, dt + \sigma(\bar{u}(t))dW_t \quad (5.42)$$

is controllable and that the condition (5.23) is satisfied. Hence, we solve for $V^{(2)}(x)$ from the quadratic terms of (5.10). Having obtained $V^{(2)}(x)$, we may then solve for $u^{(1)}(x)$ in terms of $V^{(2)}(x)$. Next, suppose $m = k + 1$, then we can always solve for $V^{(k+1)}(x)$ from equation (5.10) when eigenvalues of the corresponding differential operators are nonzero. Therefore, the k^{th} order of (5.17) is solvable for $u^{(k)}(x)$ in terms of $V^{(k+1)}(x)$ $\forall\, k > 1$.

5.4 *Stochastic Stability*

Recall the Lyapunov conditions for optimality of control in Chapter 4. Here we reexamine these conditions in context of stochastic stability of the controlled system (5.2). Lyapunov's second method for stochastic dynamics, and SDE have been studied by Mao [58], and Arnold [59]. The Barbashin-Krasovskii-LaSalle theorem for SDEs has also been studied in [60], [61].

Let us assume that the HJB associated with (5.2) has already been solved and $V(x,t)$ and $u(x)$ are known. Let us further define the infinitesimal generator acting on the $C^{2,1}$ function $V(x,t)$, as

$$\mathcal{L}\, V(x,t) = \frac{\partial V(x,t)}{\partial t} + \sum_{i=1}^{n} G_i(x,u)\frac{\partial V(x,t)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j} \tag{5.43}$$

where, $G(x,u)$, $t \geq 0$, is given by (5.2), and $a_{i,j}\,(u) = (\sigma(u)\sigma(u)^T)_{i,j}$ by (5.3). Due to Khasminskii [62], we have the following theorems regarding the solution stability of the stochastic differential equation (5.2):

Theorem 5. [62] Suppose there exist positive definite function $V(x,t) \in C^{2,1}$ satisfying

$$\mathcal{L}V(x,t) \leq 0, \text{ for } x \neq 0$$

for all $x \in \mathbb{R}^3$. Then the trivial solution $x(t) = 0$ of the SDE (5.2) is stable.

Theorem 6. [62] Let $\mathcal{K}$ denote a class of strictly increasing continuous functions $C: \mathbb{R}^+ \to \mathbb{R}^+$ such that $C(0) = 0$. Suppose there exists a positive definite function $V(x,t) \in C^{2,1}$, and $C_1, C_2, C_3 \in \mathcal{K}$ satisfying the following

$$C_1(|x|) \le V(x,t) \le C_2(|x|)$$

$$\mathcal{L}V(x) \le -C_3(|x|)$$

for all $x \in \mathbb{R}^3$. Then the trivial solution $x(t) = 0$ of the SDE (5.2) is asymptotically stable. That is, $\lim_{t\to\infty} |x_t| = 0, \forall\, x_o \in \mathbb{R}^3$.

Accordingly, if a function in $C^{2,1}$ can be found such that the conditions of Theorems 5 and 6 are satisfied, then the dynamics model (5.2) is asymptotically stable. Consider the power series solution $V^*(x)$ to equations (5.10), and (5.11) as a candidate. Since we have that

$$\mathcal{L}^* V^*(x) = -r(x) \le 0 \tag{5.44}$$

where $\mathcal{L}^*$ is given by (5.6), with argument u^* the solution to (5.11), hence, the conditions for Theorem 5 and 6 are satisfied. Then, the system (5.2) is asymptotically stable with region of attraction given by the radius of convergence of the power series V^*. If the truncated m^{th} order solution of (5.16), and (5.17) is used as the candidate, then asymptotic stability is achieved for the radius of convergence of the truncated series.

5.5 *Computation of the Stochastic Optimal Attitude Control Law*

In this section, we compute the optimal control of a spacecraft attitude control system equipped with three thruster pairs, hence let us set $m = 3$, where m is the number of available external torques. According to the results of sections 2.4, and 2.5, we assume that the system is controllable, both in a linear and nonlinear sense. That is, the three external torques are applied about three linearly independent axes. Furthermore, we assume that matrices B, Q, and R are diagonal (see section 2.2 for choosing matrix B). Specifically, we have that the linear system (5.42) is controllable (i.e. pair (A, B) is controllable) and condition (5.23) is satisfied.

Let P_i be the diagonal entries of the symmetric optimal matrix P, where $i = 1,2,3$. Similarly, let B_i, Q_i, and R_i be the diagonal elements of matrices B, Q, and R respectively. Recall that $B_i = \frac{b_i}{I_{ii}}$, where I_{ii} are the diagonal elements of I. Hence, solving (5.19) results in two solutions. Note that we will only use the positive root. This is due to the positive definite nature of the value function $V(x)$. The diagonal entries of P are obtained as

$$P_i = \frac{\varepsilon^2 B_i Q_i + \sqrt{\varepsilon^4 B_i^2 Q_i^2 + 4 R_i Q_i}}{2 B_i} \tag{5.45}$$

Note that the non-diagonal elements are zero, i.e. $P_{ij} = 0$ when $i \neq j$. Let us next, using (5.18), solve for the linear control gains K in terms of the P. We have that the resultant linear gain matrix is diagonal as well. Using the same notation, the diagonal entries, K_i, are given by

$$K_i = -\frac{B_i P_i}{R_i + \varepsilon^2 B_i^2 P_i} \tag{5.46}$$

Next, using (5.27) and (5.28) we solve for the value function expressions $V^{(3)}(x)$ and $V^{(4)}(x)$. We obtain the value function up to a quartic degree as follows

$$V(x) = \frac{1}{2} x^T P x + \phi_3 \; x_1 x_2 x_3 + \phi_{4_{12}} \; x_1^2 x_2^2 + \phi_{4_{13}} \; x_1^2 x_3^2 + \phi_{4_{23}} \; x_2^2 x_3^2 \tag{5.47}$$

where the coefficient of the 3rd order value function polynomial ϕ_3 is given by

$$\phi_3 = \frac{-I_{22}^2 \, I_{33} \, P_1 + \, I_{22} \, I_{33}^2 \, P_1 + I_{11}^2 \, I_{33} \, P_2 - I_{11} \, I_{33}^2 \, P_2 - \, I_{11}^2 \, I_{22} \, P_3 + \, I_{11} \, I_2^2 \, P_3}{b_1 \, I_2 \, I_3 \, K_1 + \, b_2 \, I_1 \, I_3 \, K_2 + \, b_3 \, I_{11} \, I_{22} \, K_3} \tag{5.48}$$

Similarly, the coefficients of the 4th order value function, ϕ_4's, are solved. Note that the optimal coefficients $\phi_{4_{12}}, \phi_{4_{13}}, \phi_{4_{13}}, \phi_{4_{23}}$ are functions of design parameter I, control gains and constants K, Q, R, P, and the noise level ε. Having obtained the optimal cost that satisfies (5.8), we solve for the corresponding optimal control degree by degree. Equations (5.46), (5.24), and (5.25) give the following truncated optimal control

$$u_1(x) = K_1\, x_1 + k_{11}\, x_2 x_3 + k_{12} x_1 x_2^2 + k_{13}\, x_1 x_3^2$$

$$u_2(x) = K_2\, x_2 + k_{21}\, x_1 x_3 + k_{22}\, x_1^2 x_2 + k_{23}\, x_2 x_3^2 \tag{5.49}$$

$$u_3(x) = K_3\, x_3 + k_{31}\, x_1 x_2 + k_{32}\, x_1^2 x_3 + k_{33}\, x_2^2 x_3$$

where K_i are the linear gains (5.46), and k_{ij}'s are the nonlinear control gains. Numerical results of this control algorithm are presented in chapter 6.

5.6 *Optimality of the Linear Control*

It is often desired to know when linear optimal control is sufficient? In this section, we answer this question for the physical model of the spacecraft attitude dynamics (5.2). We caution the reader in that the following results are for a deterministic setting. For the uncertain nonlinear system (5.2), nonlinear control is always required to achieve optimality.

Consider the second order polynomial of the value function, $V^{(2)}(x)$, for the case when $x \in \mathbb{R}^2$. Let C be the value of the function $V^{(2)}(x)$ for any $t \geq 0$

$$2V^{(2)}(x) = x^T P x = (P_1 x_1^2 + 2P_{12} x_1 x_2 + P_2 x_2^2) = C \tag{5.50}$$

For $x^T P x$, one can always find an orthogonal change of variables, $x = \beta \tilde{x}$, such that

$$x^T P x = (\beta\tilde{x})^T P(\beta\tilde{x}) = \tilde{x}^T(\beta^T P\beta)\tilde{x} = \tilde{x}^T\tilde{P}\tilde{x} = \left(\tilde{P}_1\tilde{x}_1^2 + \tilde{P}_2\tilde{x}_2^2\right) = C \tag{5.51}$$

where β is the matrix with eigenvectors of P as its columns, and $\tilde{P}_1$, $\tilde{P}_1$ are the eigenvalues of P. Then the resulting form of (5.50) is of an ellipse equation.

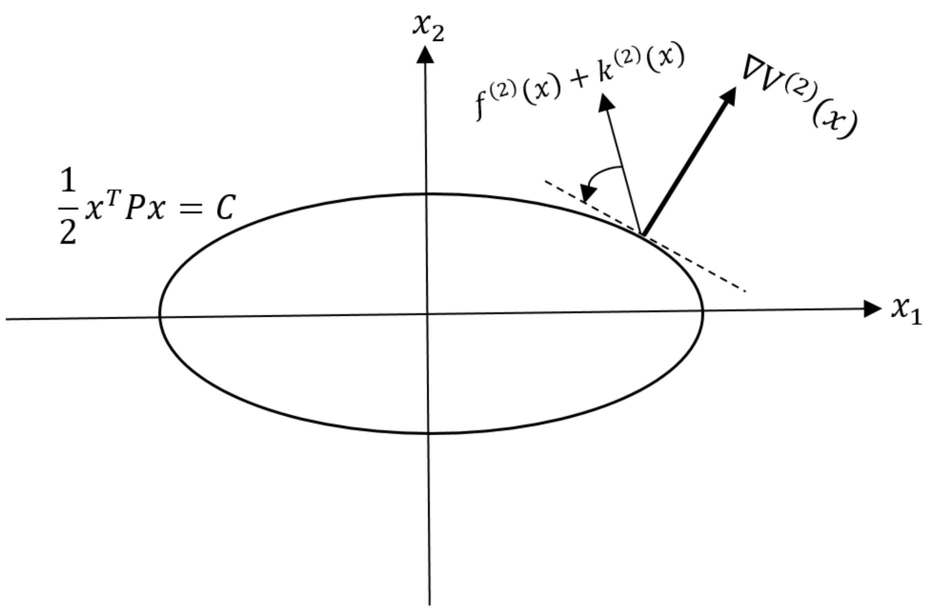

Figure 6. The Second Order Value Function Ellipse in Two Dimensions

Next, we have that the second order terms of the HJB associated with the deterministic version of system (5.2): $\dot{x} = f(x) + Bu$, have the following form, i.e. set $m = 2$ for the LHS of (4.21)

$$\left[f^{(2)}(x)^T + Bk^{(2)}(x)\right]^T \nabla V^{(2)}(x) \tag{5.52}$$

We further have that $\nabla V^{(2)}(x) = Px$, hence

$$\langle f^{(2)}(x)^T + Bk^{(2)}(x), Px\rangle \tag{5.53}$$

where $\langle .,. \rangle$ is a dot product. The matrix P is given by (5.45) where for $\varepsilon = 0$, $P_i = \frac{I_i}{P_i}\sqrt{Q_i R_i}$.

Lemma 2. Consider the deterministic system $\dot{x} = f(x) + Bu$, with the drift term given by (5.4) as $f^{(2)}(x) = I^{-1}S(x_t)Ix_t$, $x \in \mathbb{R}^3$. Suppose that arguments of P are such that $Q_1R_1 = Q_2R_2 = Q_3R_3$, and $b_1 = b_2 = b_3$, then the nonlinear control $k^{(2)}(x) = 0$.

Proof Suppose Q_i, R_i, $b_i \in \mathbb{R}$, $i = 1,2,3$, and $Q_1R_1 = Q_2R_2 = Q_3R_3$, $b_1 = b_2 = b_3$. Then, from (5.53) we have

$$\langle f^{(2)}(x)^T, Px\rangle = \begin{bmatrix} \frac{I_{22}-I_{33}}{I_{11}}x_2x_3 \\ \frac{I_{33}-I_{11}}{I_{22}}x_3x_1 \\ \frac{I_{11}-I_{22}}{I_{33}}x_1x_2 \end{bmatrix} . \begin{bmatrix} \frac{I_1}{b_1}\sqrt{Q_1R_1}x_1 & \frac{I_2}{b_2}\sqrt{Q_2R_2}x_2 & \frac{I_3}{b_3}\sqrt{Q_3R_3}x_3 \end{bmatrix}$$

$$\Rightarrow \langle f^{(2)}(x)^T, Px\rangle = \left[\frac{b_2b_3(I_{22}-I_{33})\sqrt{Q_1R_1}+b_1b_3(I_{33}-I_{11})\sqrt{Q_2R_2}+b_1b_2(I_{11}-I_{22})\sqrt{Q_3R_3}}{b_1b_2b_3}\right]x_1x_2x_3$$

$$\Rightarrow \langle f^{(2)}(x)^T, Px\rangle = \left[(I_2-I_3)\sqrt{Q_1R_1} + (I_3-I_1)\sqrt{Q_2R_2} + (I_1-I_2)\sqrt{Q_3R_3}\right]x_1x_2x_3$$

$$\Rightarrow \langle f^{(2)}(x)^T, Px\rangle = [(I_2-I_2) + (I_3-I_3) + (I_1-I_1)]\,x_1x_2x_3$$

$$\therefore \langle f^{(2)}(x)^T, Px\rangle = 0$$

This further implies that $\langle f^{(2)}(x)^T + Bk^{(2)}(x), Px\rangle = Bk^{(2)}(x)Px$. We also have that for a second order stabilizing control $k^{(2)}(x)$, the product $\langle f^{(2)}(x)^T + Bk^{(2)}(x), Px\rangle \to 0$. But we have that $\langle f^{(2)}(x)^T + Bk^{(2)}(x), Px\rangle = Bk^{(2)}(x)Px$, hence $k^{(2)}(x) = 0$. ■

In practice, the constants $b_i \in \mathbb{R}$ are chosen by design and the control system designer. From the above argument, we can conclude that choosing Q_i, R_i, $\in \mathbb{R}$ appropriately determines the effort needed by the second order controller to stabilize $f^{(2)}(x)$. In other words, for the deterministic nonlinear system with drift $f^{(2)}(x)$, a linear control can be made optimal if matrices Q, R, and B are chosen appropriately. From a geometric point of view, the value function $V^{(2)}(x) = C$, is an ellipse for $x \in \mathbb{R}^2$, and an ellipsoid for $x \in \mathbb{R}^3$. Then we have that $\nabla V^{(2)}(x)$ is a vector that is normal to the tangent plane of the ellipse $V^{(2)}(x) = C$ at all times, as shown in figure 6. Vector $f^{(2)}(x)$, lies between the normal vector $\nabla V^{(2)}(x) = Px$, and the tangent plane. In fact, if the $f^{(2)}(x)$ is not already in the tangent plane, the addition of the nonlinear feedback term $Bk^{(2)}(x)$ is tasked with driving to zero the projection of $f^{(2)}(x)$ on the $\nabla V^{(2)}(x)$ vector. In lemma 2, it was shown that if choice of Q_i,R_i,b_i and consequentlyP_i, places the $f^{(2)}(x)$ vector in the tangent plane of the ellipse, then the contribution of the nonlinear feedback $k^{(2)}(x)$ is zero.

When $f^{(2)}(x)$ lies tangent to the energy ellipsoid surface, the dynamics will not push the energy level outwards (which is what $k^{(2)}(x)$ is controlling against when $f^{(2)}(x)$ is not tangent). Linear control will keep shrinking the radius of that ellipsoid. If at some point we stop applying linear control, it will just stay at that radius/energy level; $f^{(2)}(x)$ won't make it grow if it's tangent to the surface. If we keep applying linear control, it will keep shrinking the energy ellipsoid and since $f^{(2)}(x)$ won't destabilize that process, linear control is all that's needed to drive energy (and hence velocity) to zero.

A similar, but more general results were obtained by Ikeda and Šiljak [63], where they showed that for a specific form of a running cost function, the linear control can be made optimal. It was proven that if the cost function has the form

$$r(t, x, u) = x^T Q x - 2x^T P f(t, x, u) + u^T R u$$

where f is a nonlinear vector field, and r is the running cost, then the linear control is optimal. In comparison to these results, in lemma 2 we've shown that the product $2x^T Pf(t, x, u)$ is zero.

However, it is important to point out that the linear control can be made optimal only when the dynamics are deterministic. In a nonlinear stochastic setting, such as the dynamics (5.2), the uncertainty effects carry into the higher orders of the HJB. Hence, nonlinear control is needed when considering the optimality of nonlinear system with multiplicative control noise.

5.7 *Effects of Physical Parameters on Control Gains*

Often the inputted physical parameters affect the behavior of a feedback control system. Specifically, for the feedback controller of section 5.5, the noise level ε affects the control gains. In this section, the effects of moment of inertia I, and the noise level: ε variation on the linear gain matrix K is studied. Suppose that the physical parameters and the control constants are arbitrarily chosen as $I_i = 0.4$, $b_i = 1$, and $R_i = Q_i = 1$, where the subscript i denotes the i^{th}axis. Then varying the values of $\varepsilon \in [0,1]$, we generate the array of values corresponding to the different control gain values K_i. Figure 7 describes the effects of various noise levels on the linear control.

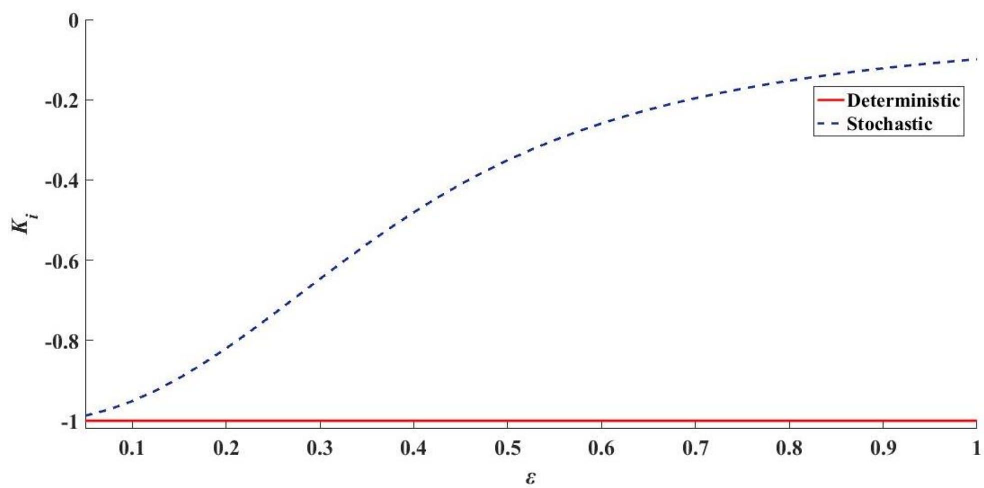

Figure 7. Effects of Varying the Noise level on Linear Control Gain

As shown on the plot, a deterministic controller does not react to varying noise levels. On the other hand, a stochastic control will lower the magnitude of the gain as the noise level is increased. This translates to the situation where highly uncertain thrust output is suppressed through application of small magnitude thrust. In general, note that for $\varepsilon = 0$, the linear stochastic control becomes a linear deterministic control, and the nonlinear stochastic control becomes a nonlinear deterministic control.

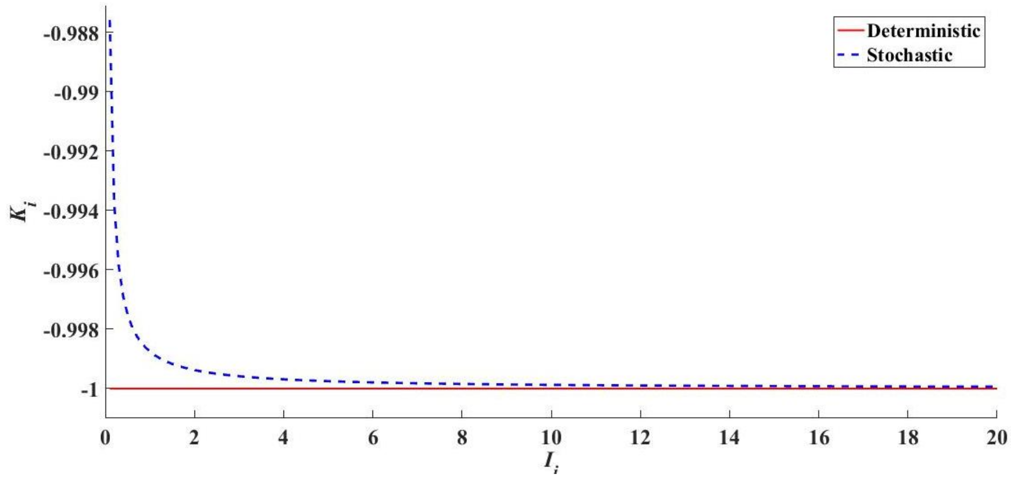

Figure 8. Effects of Varying the Moment of Inertia on Linear Control Gain

While varying the noise percentage affects the linear gain matrix, varying the mass moment of inertia affects the magnitude of the linear control gain (5.46) as well. Suppose that $\varepsilon = 0.015$ is chosen arbitrarily. Figure 8 shows that a lower mass moment of inertia in design of a spacecraft results in a lower magnitude of the stochastic control gain. As the moment of inertia of the i^{th} axis is increased, the behavior of the linear stochastic control gain converges to that of a linear deterministic controller for the assumed diagonal form of the matrices of section 5.5.

CHAPTER 6
Numerical Experiments

6.1 *Detumbling of a 6U CubeSat*

Throughout this chapter, we will assume that the spacecraft model is a 6U CubeSat with three thruster pairs. For a 6U CubeSat, the standard dimensions are given as $10 \times 20 \times 30$ centimeters, and the maximum mass is 6 kg. Using table 8, the entries of moment of inertia tensor in principal axes (in units of kilogram meter squared) are calculated as follows:

$$I_{11} = 0.05\,,\ I_{22} = 0.065,\ I_{33} = 0.025 \tag{6.1}$$

We also assume that the thrusters are installed symmetrically, and the three torque axes are

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.2}$$

The goal is to compare the performance and optimality results of the stochastic nonlinear control derived in section 5.5, to a linear deterministic controller for a CubeSat with thrust uncertainty. We shall consider two cases of $\varepsilon = 0.14$ and $\varepsilon = 0.28$. i.e. the uncertainty has standard deviation of 14%, and 28% from the nominal thrust, for $\varepsilon = 0.14$, and $\varepsilon = 0.28$ respectively. The control gains, similar to other parameters, have been kept the same for both controllers. We have trivially chosen the control gains as

$$Q = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.3}$$

Note that the input vectors (6.2), along with the choice of nonlinear gains (6.3) satisfy the conditions of the section 5.6 on optimality of the Linear control. This means with no thrust uncertainty, the linear and nonlinear controllers will perform identically. However, in this chapter we demonstrate the superiority of stochastic nonlinear control when the generated thrust contains uncertainty.

For the experiments of this chapter, we have randomly generated a list of 50 random initial conditions that are within norm 1 of the origin. A second group of 50 randomly generated initial conditions are also selected between norm 1 to norm 3 of the origin. For proof of the numerical results, each set of 50 initial conditions is shifted to 8 different octants around the origin. This done by creating all the negative and positive combinations of coordinates $|x_1|$, $|x_2|$, and $|x_3|$ of angular rate. We shall assign the following designations to each octant

Table 1. Assignment of Initial Conditions into Octants

Region I: (x_1, x_2, x_3)	Region II: $(-x_1, x_2, x_3)$	Region III: $(x_1, -x_2, x_3)$	Region IV: $(x_1, x_2, -x_3)$,
Region V: $(-x_1, -x_2, x_3)$	Region VI: $(x_1, -x_2, -x_3)$,	Region VII: $(-x_1, x_2, -x_3)$	Region VIII: $(-x_1, -x_2, -x_3)$

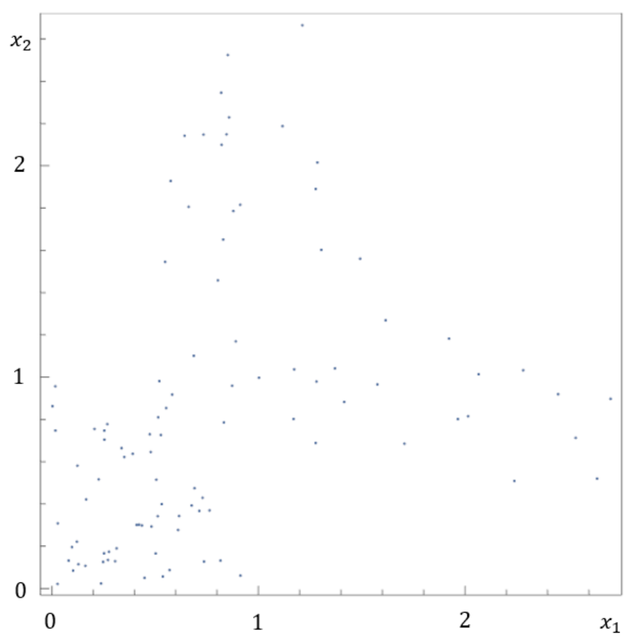

Figure 9. Randomly Generated Initial Conditions of the First Octant

The list of generated initial conditions of the first octant is shown here. Note that the list is in the ascending order of distance to the origin. The initial conditions of the rest of the Octants are generated from this list.

Table 2. Table of Randomly Generated Initial Condition Coordinates

	$Norm$	$\lvert x_1\rvert$	$\lvert x_2\rvert$	$\lvert x_3\rvert$		$Norm$	$\lvert x_1\rvert$	$\lvert x_2\rvert$	$\lvert x_3\rvert$
1	0.15746	0.07899	0.13429	0.02286	**51**	1.16897	0.50326	0.51749	0.91946
2	0.29319	0.09499	0.19843	0.19381	**52**	1.32920	0.51785	0.98371	0.72863
3	0.31622	0.26912	0.13726	0.09342	**53**	1.65226	0.55097	0.85658	1.30103
4	0.33126	0.10050	0.08741	0.30330	**54**	1.73205	1.00000	1.00000	1.00000
5	0.36545	0.11903	0.22401	0.26307	**55**	1.76096	0.80142	1.46083	0.56981
6	0.38857	0.24593	0.12827	0.27213	**56**	1.82052	1.36772	1.04407	0.59460
7	0.41468	0.30405	0.13154	0.24942	**57**	1.82853	0.54593	1.54826	0.80521
8	0.50791	0.02631	0.31068	0.40094	**58**	1.89372	0.68459	1.10392	1.37799
9	0.50872	0.40801	0.30344	0.01590	**59**	1.90042	1.41264	0.88504	0.91255
10	0.56404	0.48015	0.29591	0.00519	**60**	1.96964	0.83031	0.78834	1.60268
11	0.56892	0.16377	0.42434	0.34174	**61**	1.97633	1.70431	0.68774	0.72680
12	0.62199	0.15979	0.10974	0.59102	**62**	2.00291	1.27849	0.98193	1.18867
13	0.62211	0.51012	0.34465	0.08954	**63**	2.09470	0.57249	1.93118	0.57495
14	0.65285	0.56736	0.08971	0.31026	**64**	2.11221	0.58017	0.92042	1.81043
15	0.67303	0.25016	0.16916	0.60148	**65**	2.14225	0.90931	1.81868	0.67437
16	0.67435	0.22464	0.51890	0.36748	**66**	2.16183	0.88798	1.17203	1.58472
17	0.68376	0.27437	0.17632	0.60097	**67**	2.20962	0.52536	0.72920	2.01859
18	0.68474	0.44626	0.05272	0.51667	**68**	2.23254	1.96285	0.80398	0.69648
19	0.72829	0.12122	0.58347	0.41866	**69**	2.23485	1.57311	0.96871	1.25758
20	0.75386	0.33501	0.66673	0.10747	**70**	2.31562	0.65993	1.80884	1.28635
21	0.75397	0.61377	0.34554	0.26900	**71**	2.38679	0.84321	2.15181	0.59622
22	0.76929	0.41927	0.30497	0.56835	**72**	2.40175	1.92028	1.18493	0.82271
23	0.78167	0.49985	0.16868	0.57680	**73**	2.44536	0.82686	1.65370	1.60042
24	0.80521	0.47640	0.64844	0.03058	**74**	2.46671	1.16748	0.80469	2.01844
25	0.81448	0.67441	0.39521	0.22882	**75**	2.47979	0.87041	0.96191	2.11340
26	0.81759	0.02512	0.02396	0.81685	**76**	2.49884	1.30189	1.60502	1.40471
27	0.82075	0.34828	0.62454	0.40284	**77**	2.52209	0.87609	1.78874	1.54721
28	0.83341	0.71110	0.37008	0.22790	**78**	2.55173	2.06351	1.01668	1.10436
29	0.83454	0.26592	0.78062	0.12792	**79**	2.62017	0.51235	0.81271	2.43768
30	0.84145	0.01488	0.75000	0.38122	**80**	2.64979	1.61323	1.27147	1.67399
31	0.87624	0.53492	0.05924	0.69148	**81**	2.67490	2.44861	0.92203	0.55613
32	0.87893	0.00060	0.86554	0.15287	**82**	2.69200	0.73165	2.15080	1.44418
33	0.88060	0.25208	0.70694	0.46060	**83**	2.70173	1.27497	1.89324	1.44547
34	0.88188	0.12590	0.11716	0.86495	**84**	2.71476	1.11441	2.19045	1.15324
35	0.88651	0.73416	0.12997	0.47961	**85**	2.74945	2.27883	1.03510	1.13797
36	0.89410	0.23599	0.02621	0.86200	**86**	2.79516	1.28284	2.01838	1.44685
37	0.90999	0.52948	0.40196	0.62141	**87**	2.80883	2.53310	0.71545	0.98034
38	0.92952	0.72739	0.43175	0.38538	**88**	2.81381	0.85542	2.23230	1.48412
39	0.93310	0.68824	0.47702	0.41166	**89**	2.84890	1.27485	0.69119	2.45219
40	0.93320	0.91036	0.06357	0.19511	**90**	2.85161	0.81942	2.10226	1.74377

Table 2 (cont.). Table of Randomly Generated Initial Condition Coordinates

	$Norm$	$\|x_1\|$	$\|x_2\|$	$\|x_3\|$		$Norm$	$\|x_1\|$	$\|x_2\|$	$\|x_3\|$
41	0.94243	0.76086	0.37189	0.41347	**91**	2.87071	2.63723	0.52254	1.00646
42	0.96491	0.31098	0.19244	0.89292	**92**	2.88035	1.17075	1.03981	2.41755
43	0.96771	0.47161	0.73240	0.42146	**93**	2.89653	0.84955	2.52713	1.13215
44	0.97513	0.01435	0.95928	0.17450	**94**	2.93822	2.70196	0.89995	0.72294
45	0.98013	0.81398	0.13422	0.52922	**95**	2.94763	2.23661	0.51148	1.85055
46	0.98393	0.38872	0.63969	0.63860	**96**	2.95110	2.01250	0.81755	1.99761
47	0.99343	0.43344	0.30100	0.84168	**97**	2.96206	0.64021	2.14525	1.93955
48	0.99366	0.60879	0.27936	0.73396	**98**	2.96453	0.81685	2.34852	1.61420
49	0.99450	0.25164	0.74966	0.60309	**99**	2.96781	1.48976	1.56312	2.03597
50	0.99884	0.20397	0.75757	0.61819	**100**	2.98571	1.21029	2.66808	0.57535

Next, to inspect the trajectories of the system during a detumbling maneuver under thrust uncertainty, we select two coordinates with considerable difference in distance from the origin. For demonstration, let us select the coordinate numbers 20, and 90. We would like to inspect the trajectories of the system with trivial gains (6.3), in Region I, for the case when $\varepsilon = 0.28$. The following trajectories are simulated for 5 realizations:

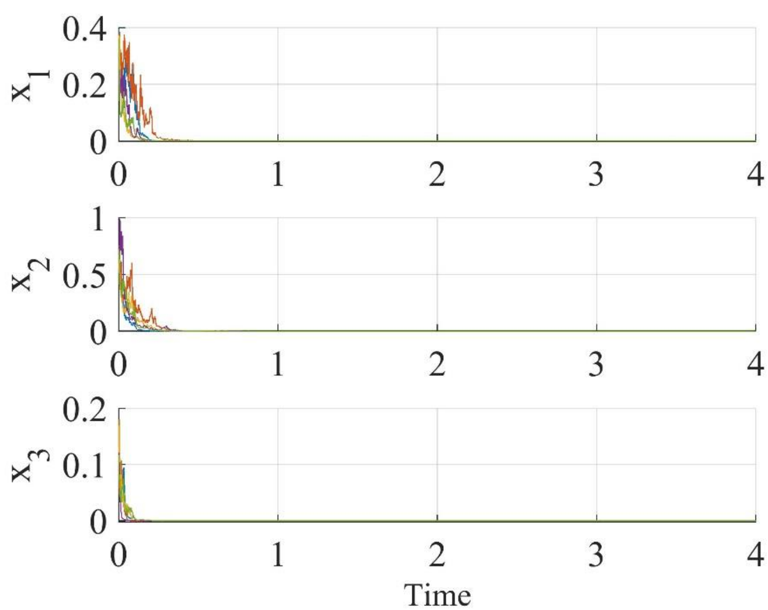

Figure 10. Stochastic Nonlinear Controller State Trajectory in Region I - Coordinate 20

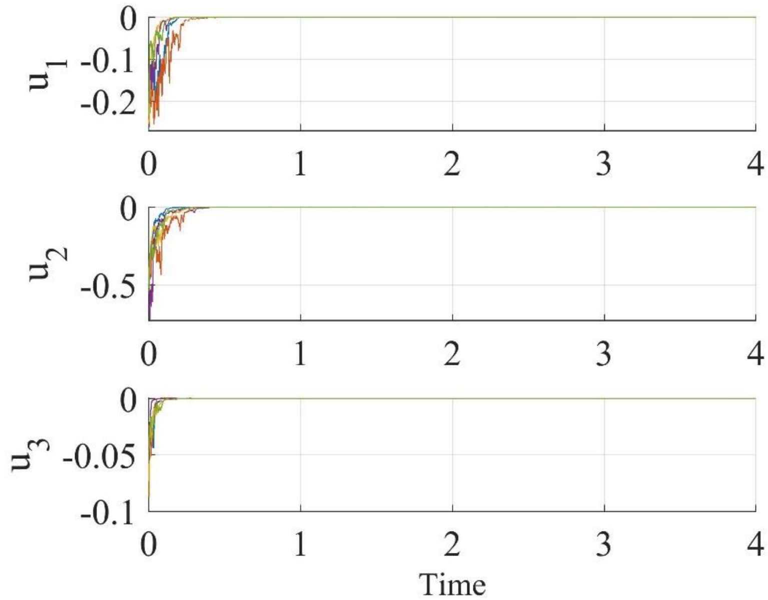

Figure 11. Stochastic Nonlinear Controller Control Trajectory in Region I - Coordinate 20

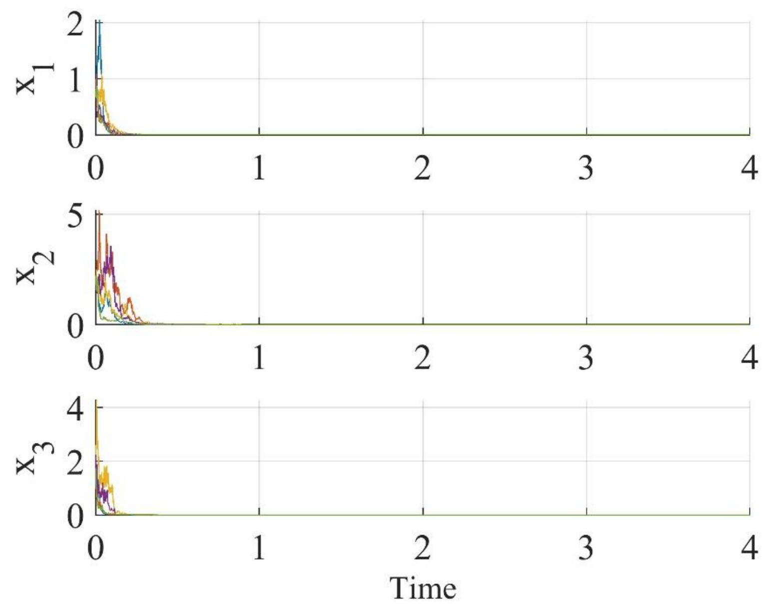

Figure 12. Stochastic Nonlinear Controller State Trajectory in Region I - Coordinate 90

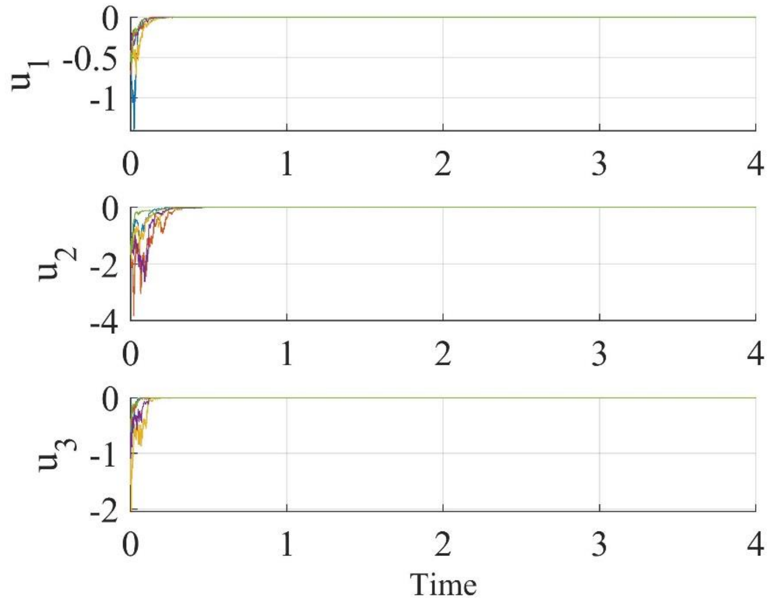

Figure 13. Stochastic Nonlinear Controller Control Trajectory in Region I - Coordinate 90

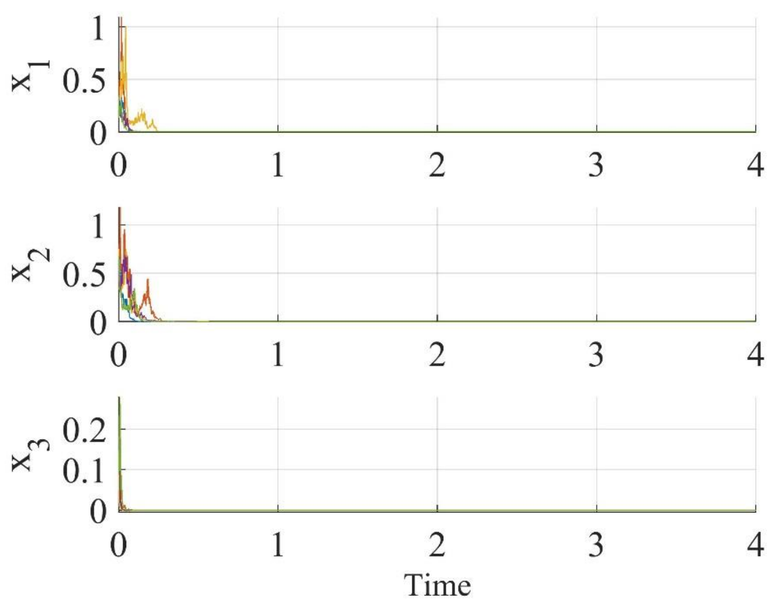

Figure 14. Deterministic Linear Controller State Trajectory in Region I - Coordinate 20

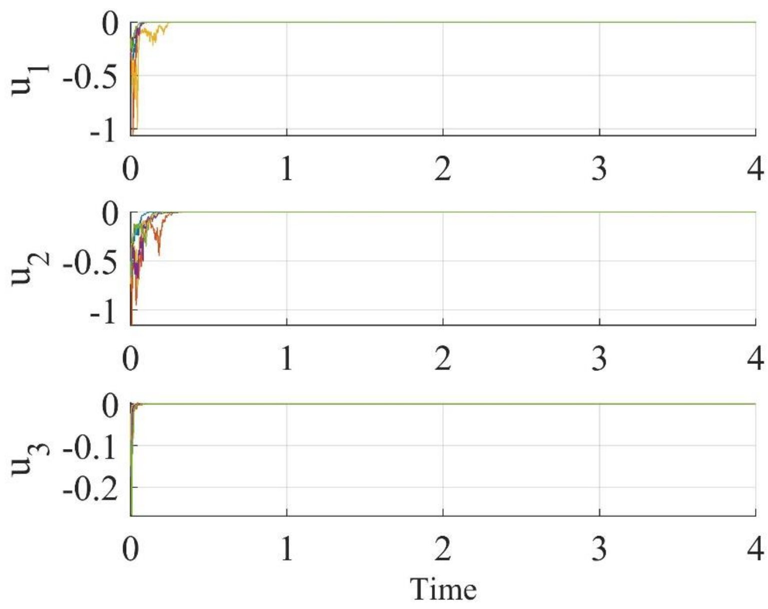

Figure 15. Deterministic Linear Controller Control Trajectory in Region I - Coordinate 20

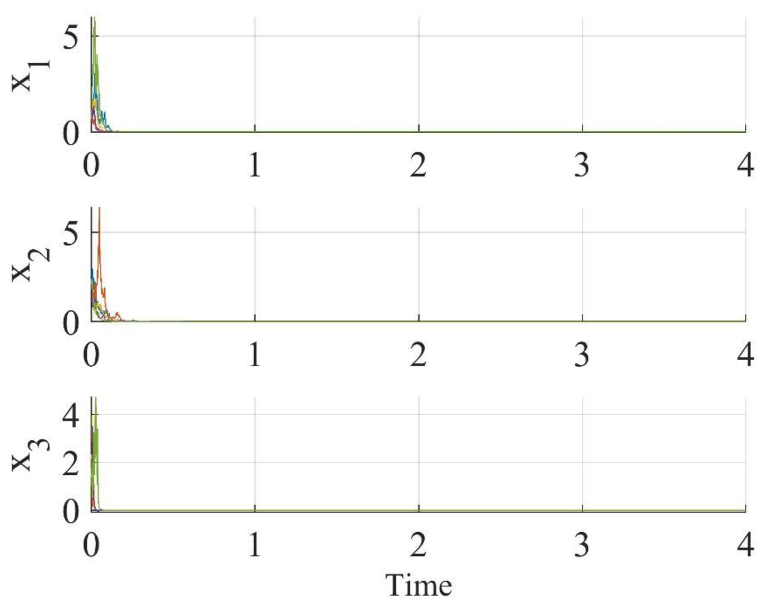

Figure 16. Deterministic Linear Controller State Trajectory in Region I - Coordinate 90

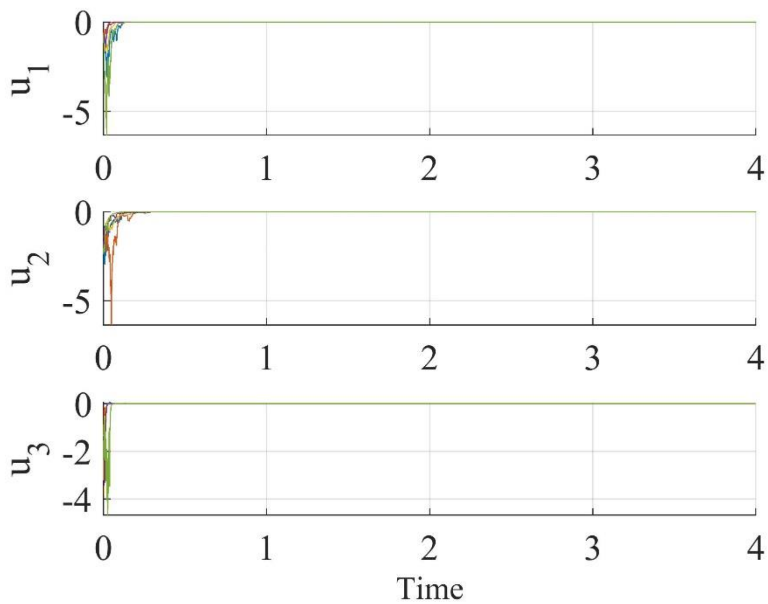

Figure 17. Deterministic Linear Controller Control Trajectory in Region I - Coordinate 90

Let us now compare the trajectories of the coordinate number 90 in Region I, with $\varepsilon = 0.28$, with the following control gains:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1/I_1^2 & 0 & 0 \\ 0 & 1/I_2^2 & 0 \\ 0 & 0 & 1/I_3^2 \end{bmatrix}$$

Clearly, the choice of gains has penalized the control input (and hence the thrust magnitude). This means that the thrusters are commanded to actuate with a smaller thrust magnitude, thus, increasing the settling time of the five realizations. We have demonstrated this strategy as a beneficial method to both reduce the propagated uncertainty, as well as, reducing the fuel consumption. Though, the disadvantage of such control strategy is the longer settling times and increased variations among the 5 realizations compared to the previous simulated trajectory of the coordinate 90 with the trivial gains (6.3). The following two plots are the trajectories of the nonlinear stochastic control, and the linear deterministic control with the $R_i = 1/I_i^2$, $i = 1,2,3$.

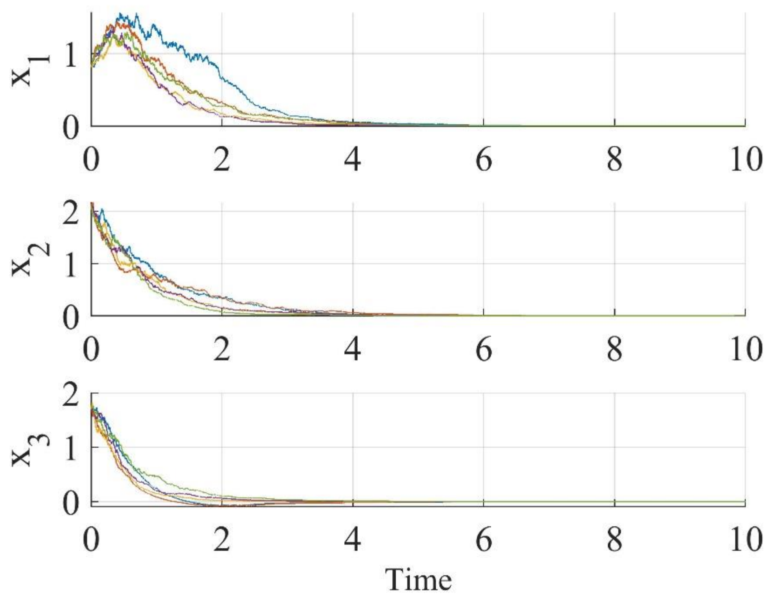

Figure 18. Stochastic Nonlinear Controller Control Trajectory in Region I - Coordinate 90

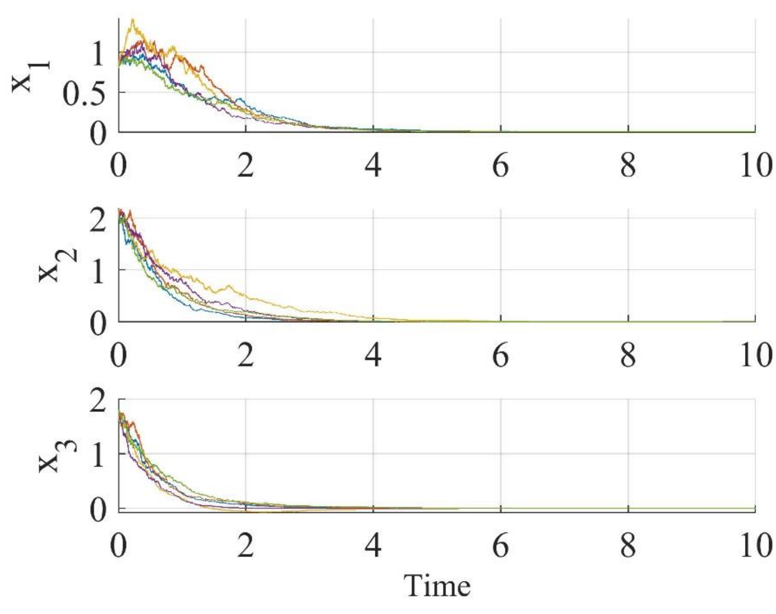

Figure 19. Deterministic Linear Controller State Trajectory in Region I - Coordinate 90

6.2 *Monte Carlo Results*

In this section, the performance of section's 5.5 stochastic optimal controller is compared to that of a deterministic optimal controller applied to stochastic dynamics through an Monte Carlo experiment performed over the 8 regions defined in Table 1. The tables show the mean cost of the set of initial conditions given in Table 2. Each initial condition is simulated for 2000 realizations with stochastic nonlinear and deterministic linear controls. This experiment is performed for values of $\varepsilon = 0.14$, and $\varepsilon = 0.28$. The Initial conditions of Region I are displayed in this section. Data corresponding to the Regions II-VIII is tabulated in Appendix B.

Table 3. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region I and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
1	0.157460926	0.001787112	0.001756035	1.738940351
2	0.293187079	0.005037878	0.004920388	2.33213718
3	0.316221076	0.006311412	0.0060878	3.542977192
4	0.33126114	0.005101999	0.004662632	8.611661092
5	0.365447776	0.007284514	0.00737724	-1.272922766
6	0.388573818	0.008131774	0.00750104	7.756422708
7	0.414682564	0.009815822	0.009198391	6.290166693
8	0.507910302	0.013482154	0.013781099	-2.217341664
9	0.508723927	0.017571308	0.01743703	0.764188787
10	0.564035602	0.020880404	0.020954155	-0.353204254
11	0.568921476	0.020557525	0.019734541	4.003320784
12	0.621993608	0.01865852	0.015146834	18.82081288
13	0.622111707	0.025891535	0.025585309	1.182726688
14	0.652850918	0.023042072	0.023932802	-3.865666935
15	0.67303375	0.021142996	0.018779783	11.17728404
16	0.674354319	0.029970584	0.029152984	2.728006034
17	0.683764431	0.021077498	0.020672257	1.922623535
18	0.684739975	0.022408794	0.022692774	-1.267268872
19	0.728289287	0.034512863	0.033811517	2.032128611
20	0.753863006	0.04262121	0.040998754	3.806687817

Table 3 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region I and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
21	0.753969075	0.035947219	0.034101678	5.134030623
22	0.769293241	0.032891125	0.0299939	8.808531349
23	0.781665858	0.03182285	0.029936885	5.926451621
24	0.805212622	0.046005618	0.045666789	0.736494942
25	0.814479069	0.042395205	0.042805415	-0.967585043
26	0.817586623	0.02630429	0.023865562	9.271215112
27	0.820748706	0.044133785	0.043270799	1.955386346
28	0.833407694	0.042913722	0.043080878	-0.3895156
29	0.834535397	0.051402634	0.051255261	0.286704836
30	0.841453615	0.049812823	0.049420447	0.787700048
31	0.876241566	0.040240983	0.036699145	8.801569929
32	0.87893391	0.058238051	0.05616268	3.563599079
33	0.880603177	0.04989527	0.05018582	-0.582320692
34	0.881879127	0.031929159	0.032982955	-3.300417616
35	0.886511581	0.045672404	0.042879506	6.115067977
36	0.894100586	0.036650603	0.030782172	16.01182656
37	0.909989643	0.046288952	0.044261995	4.378923414
38	0.92952342	0.051760801	0.053652673	-3.655027316
39	0.933100196	0.053189866	0.052229509	1.805525512
40	0.933204146	0.054676087	0.052767209	3.491248693
41	0.942425394	0.053969841	0.051481763	4.610126959
42	0.964908062	0.043131534	0.039325133	8.825099542
43	0.967711414	0.062165861	0.061383382	1.258694961
44	0.975126961	0.070327167	0.072373606	-2.909884475
45	0.980126841	0.056259329	0.052075052	7.437480215
46	0.983925319	0.057841676	0.056535269	2.258591583
47	0.993427783	0.045246154	0.046325056	-2.384517571
48	0.993663086	0.052165783	0.047989905	8.005013569
49	0.994500751	0.063419471	0.062259916	1.828390743
50	0.998840156	0.062108323	0.060126635	3.190696826
			Average Cost Difference (%):	**3.36065564**

Table 4. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region I and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
51	1.168965631	0.070191713	0.068150652	2.9078377
52	1.329198658	0.117361029	0.109460345	6.731948643
53	1.652261099	0.143674911	0.139235204	3.090106102
54	1.732050808	0.179303516	0.173288531	3.354638773
55	1.760957841	0.214096262	0.211875683	1.037187152
56	1.820518877	0.217084647	0.209392151	3.543546722
57	1.828533372	0.229761775	0.223829047	2.582121236
58	1.893715218	0.204129525	0.191760085	6.059603656
59	1.900420695	0.219215456	0.216525443	1.227109244
60	1.969644102	0.20751687	0.187530226	9.631334716
61	1.976334082	0.233608943	0.231516263	0.895804623
62	2.002910229	0.228263976	0.230335222	-0.907390959
63	2.094697427	0.323792476	0.320701762	0.954535557
64	2.112206523	0.233609617	0.217494183	6.898446109
65	2.142249774	0.328475928	0.319032331	2.87497391
66	2.161830466	0.255047446	0.243609678	4.484564972
67	2.209623524	0.23657631	0.216357605	8.546377989
68	2.232543354	0.323913437	0.302221315	6.696888779
69	2.234850662	0.290089866	0.28477596	1.831813683
70	2.315624628	0.333838442	0.334374034	-0.160434653
71	2.386793676	0.407853192	0.413731914	-1.441381919
72	2.401746454	0.370879516	0.35299893	4.821130554
73	2.445361846	0.363535147	0.350316715	3.636080996
74	2.466706602	0.288637436	0.286798041	0.637268558
75	2.479790488	0.303081871	0.285979794	5.642725251
76	2.498843498	0.389289977	0.375051423	3.657570315
77	2.522092914	0.382799393	0.387668618	-1.272004341
78	2.551730261	0.419035057	0.385241011	8.064730043
79	2.620165462	0.342699584	0.281481046	17.86361591
80	2.649793028	0.390258268	0.393872377	-0.926081435
81	2.674902192	0.445186068	0.430648386	3.265529415
82	2.692003392	0.468286689	0.471840375	-0.758869871
83	2.701726462	0.491347485	0.446572784	9.112634549
84	2.714761477	0.501169848	0.497433609	0.745503644
85	2.749451218	0.455737995	0.450722332	1.10055848
86	2.795156227	0.49244978	0.485167592	1.478767706
87	2.808828434	0.468940135	0.467790329	0.245192429

Table 4 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region I and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	0.519508891	0.511291074	1.581843384
89	2.848903585	0.378647326	0.348047163	8.081441586
90	2.851611437	0.507423167	0.489537912	3.524721758
91	2.870713551	0.500022338	0.496462186	0.711998444
92	2.880350027	0.40937772	0.376648086	7.994971868
93	2.896534118	0.589656375	0.58144814	1.392036967
94	2.938218496	0.55303249	0.527127318	4.684204344
95	2.947634879	0.476586988	0.451252678	5.315778801
96	2.95110354	0.456732073	0.439179763	3.843021138
97	2.962064987	0.53606848	0.527688006	1.56332165
98	2.96452904	0.571488323	0.557432346	2.459538829
99	2.967810632	0.499924199	0.466361881	6.713481384
100	2.985712699	0.645541065	0.654191947	-1.340097886
			Average Cost Difference (%):	**3.49360493**

Table 5. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region I and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.096651786	0.081547995	15.62701662
2	0.293187079	0.154616239	0.102200972	33.9002341
3	0.316221076	0.137090686	0.059161045	56.84532147
4	0.33126114	0.157734073	0.07853505	50.21047213
5	0.365447776	0.070920477	0.041553911	41.40773901
6	0.388573818	0.116051631	0.059718463	48.54146981
7	0.414682564	0.089801276	0.046095066	48.66992116
8	0.507910302	0.018187926	0.012794015	29.65655186
9	0.508723927	0.061879376	0.042537097	31.25803803
10	0.564035602	0.203973147	0.195471202	4.16816865
11	0.568921476	0.099776774	0.081304137	18.51396426
12	0.621993608	0.146675045	0.070940608	51.63416638
13	0.622111707	0.122711451	0.069471373	43.38639741
14	0.652850918	0.13713821	0.035395664	74.18978703
15	0.67303375	0.025155954	0.009789764	61.08371046

Table 5 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region I and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.135914651	0.059152943	56.47787627
17	0.683764431	0.170392442	0.265874664	-56.03665316
18	0.684739975	0.175614766	0.100170673	42.95999383
19	0.728289287	0.131923888	0.109212637	17.21542014
20	0.753863006	0.07955313	0.041483741	47.85404237
21	0.753969075	0.044473255	0.028064927	36.8948222
22	0.769293241	0.00349229	0.003264837	6.513001516
23	0.781665858	0.056667494	0.050507944	10.86963571
24	0.805212622	0.107399856	0.105381466	1.879322671
25	0.814479069	0.136041409	0.051932838	61.82571286
26	0.817586623	0.273472699	0.087993397	67.82369961
27	0.820748706	0.184834968	0.104732662	43.33720352
28	0.833407694	0.091202424	0.059648957	34.5971803
29	0.834535397	0.196128032	0.099026417	49.50929973
30	0.841453615	0.106568052	0.082599594	22.49122271
31	0.876241566	0.2178001	0.192140617	11.78120802
32	0.87893391	0.714279515	0.04448058	93.77266475
33	0.880603177	0.212187311	0.097421455	54.08704925
34	0.881879127	0.109823904	0.091837062	16.37789332
35	0.886511581	0.015058367	0.008467066	43.77168287
36	0.894100586	0.097665928	0.104537157	-7.035441409
37	0.909989643	0.129168257	0.082245985	36.32647331
38	0.92952342	0.202160995	0.110781005	45.2015931
39	0.933100196	0.132275	0.090425568	31.63820246
40	0.933204146	0.117986472	0.084291419	28.55840345
41	0.942425394	0.230396307	0.088509791	61.58367611
42	0.964908062	2.032163585	0.072062132	96.45392071
43	0.967711414	0.191886474	0.032275764	83.17976101
44	0.975126961	0.101273018	0.089838269	11.29101214
45	0.980126841	0.08890041	0.037000485	58.37984876
46	0.983925319	0.212591783	0.099426581	53.23122097
47	0.993427783	0.044570523	0.018050496	59.50126989
48	0.993663086	0.026858774	0.014510821	45.97362831
49	0.994500751	0.076017819	0.07290567	4.093973022
50	0.998840156	0.013889551	0.010672955	23.15838909
			Average Cost Difference (%):	**38.09262336**

Table 6. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region I and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.199655741	0.246074451	-23.24937419
52	1.329198658	1.629061571	0.20705453	87.28995062
53	1.652261099	0.560073879	0.27100742	51.61220156
54	1.732050808	0.457472902	0.316354062	30.84747517
55	1.760957841	0.538337897	0.338849036	37.05644022
56	1.820518877	0.587657188	0.425581972	27.57989182
57	1.828533372	0.553294861	0.355852233	35.6848838
58	1.893715218	0.797735588	0.34855268	56.30724202
59	1.900420695	0.758826153	0.373260042	50.81086223
60	1.969644102	0.614058933	0.383024453	37.6241543
61	1.976334082	0.590945736	0.417074613	29.42251928
62	2.002910229	0.55477491	0.4282052	22.814606
63	2.094697427	0.717065336	0.505872474	29.45238764
64	2.112206523	0.700234576	0.580405351	17.11272616
65	2.142249774	1.666815002	0.523829775	68.57301055
66	2.161830466	2.319469947	0.507338008	78.12698504
67	2.209623524	2.902292244	0.420631854	85.50690906
68	2.232543354	1.071572667	0.607187502	43.33678712
69	2.234850662	0.846603261	0.598583155	29.29590722
70	2.315624628	1.227568736	0.585091523	52.33737179
71	2.386793676	1.841384102	0.670151762	63.60608516
72	2.401746454	1.223585099	0.625135675	48.90950575
73	2.445361846	1.830570116	0.696016397	61.97816237
74	2.466706602	0.82745518	0.538976265	34.86338862
75	2.479790488	0.954363069	0.617235241	35.32490303
76	2.498843498	1.194903532	0.674157	43.58063376
77	2.522092914	0.991574364	0.627608887	36.70581753
78	2.551730261	1.652615865	0.71634561	56.65383438
79	2.620165462	12.17892848	0.790493391	93.50933547
80	2.649793028	1.217476411	0.751728818	38.25516356
81	2.674902192	1.30109964	0.871426755	33.02382628
82	2.692003392	1.287323589	0.8295821	35.55760902
83	2.701726462	1.588507231	0.752006334	52.65955867
84	2.714761477	1.594900853	0.840232453	47.31757456
85	2.749451218	1.323251979	0.757936669	42.72166747
86	2.795156227	1.38144323	0.841151363	39.1106819
87	2.808828434	1.289507533	1.002050283	22.29201795

Table 6 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region I and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	1.514179775	0.834135018	44.91175805
89	2.848903585	1.606660278	0.731160186	54.49192366
90	2.851611437	1.21943244	0.911433243	25.25758601
91	2.870713551	1.582247743	0.904632217	42.82613324
92	2.880350027	3.082652688	0.803151131	73.9461038
93	2.896534118	1.271137863	1.25640027	1.159401628
94	2.938218496	1.396067091	0.868626555	37.78045763
95	2.947634879	1.822407132	1.553266664	14.7684051
96	2.95110354	2.400388075	0.964740009	59.80899843
97	2.962064987	1.329361025	1.488073413	-11.93899815
98	2.96452904	2.474384054	0.918501219	62.87960157
99	2.967810632	1.2628239	0.874121328	30.7804257
100	2.985712699	1.290949916	0.968000501	25.01641712
			Average Cost Difference (%):	**41.90601833**

The mean comparisons in Tables 3-6 show that the stochastic nonlinear control results in lower cost of the optimal control problem compared to using the deterministic linear control. Even though in cases where $\varepsilon = 0.14$ the stochastic controller had yielded a lower cost by ~3.5%. In fact, the cost difference is increased up to 50% as shown in the tables of Appendix B.

Comparing the two given initial conditions, the cost difference is increased as the distance from the initial condition to the origin is increased. From the displayed tables, this relation can also be seen when ε is increased. This shows that under smaller norms of initial conditions, the two controllers behave similarly. However, as the distance from the origin is increased, the stochastic control's mean cost decreases. This relation also follows when the noise effects are increased. The stochastic control's mean cost decreases, as ε is increased. Next, we will display the Cumulative Distribution Functions (CDF), and the approximate Probability Density Functions (PDF) of the Coordinates numbers 20 and 90, which trajectories were simulated in section 6.1.

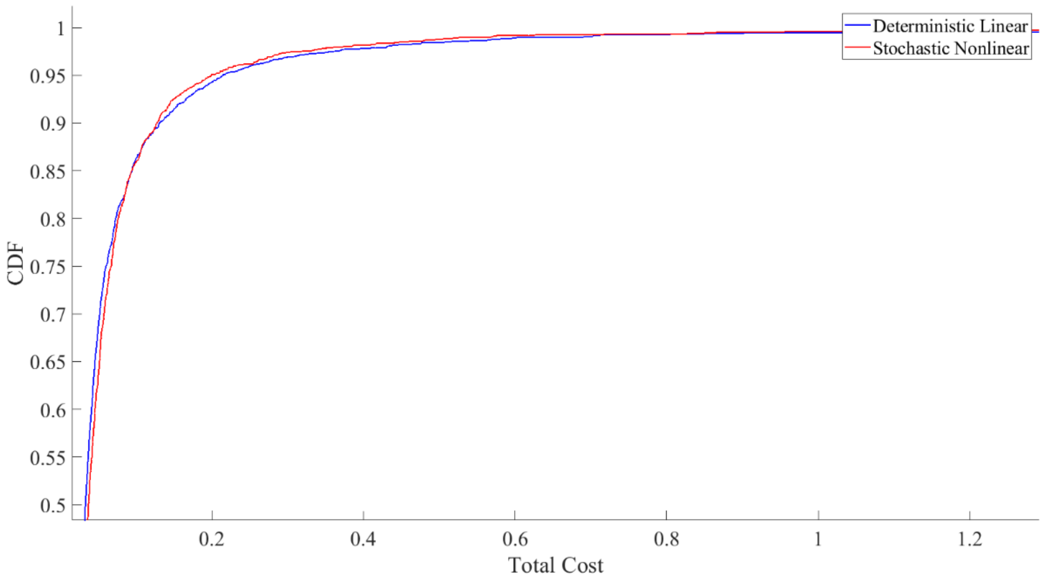

Figure 20. CDF of total cost for stochastic nonlinear control and deterministic linear control for $\varepsilon = 0.28$ - Region I - Coordinate 20

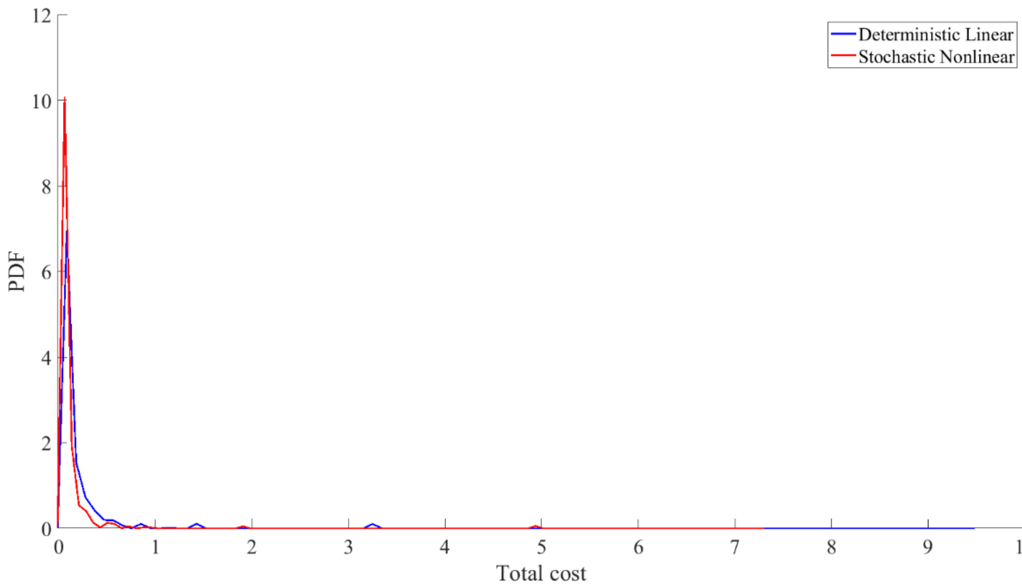

Figure 21. PDF of total cost for stochastic nonlinear control and deterministic linear control for $\varepsilon = 0.28$ - Region I - Coordinate 20

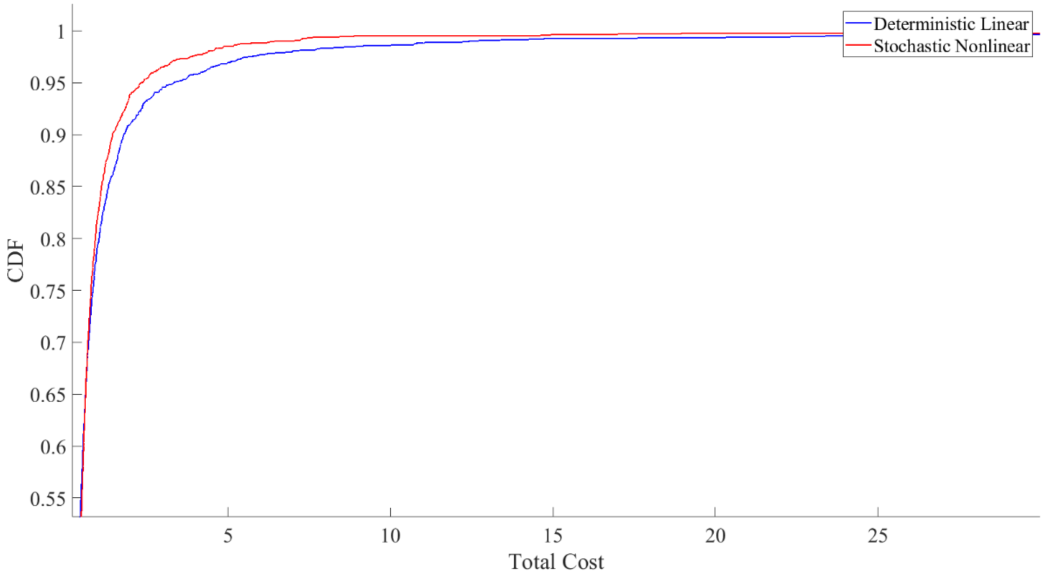

Figure 22. CDF of total cost for stochastic nonlinear control and deterministic linear control for $\boldsymbol{\varepsilon} = 0.28$ - Region I - Coordinate 90

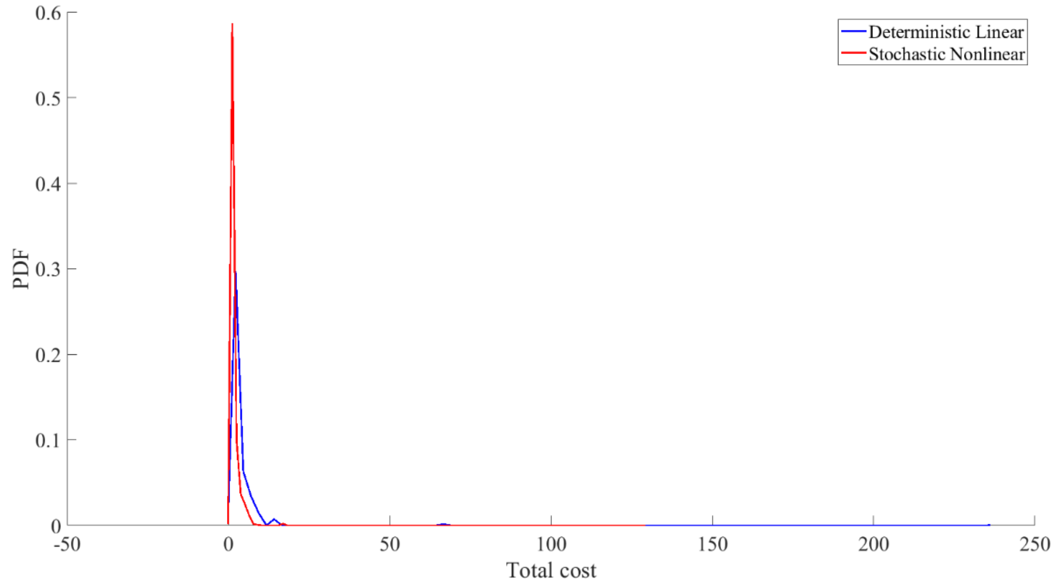

Figure 23. PDF of total cost for stochastic nonlinear control and deterministic linear control for $\boldsymbol{\varepsilon} = 0.28$ - Region I - Coordinate 90

The probability density functions (PDFs) of the total cost in figures 21, and 23 are approximated using the kernel density estimation function in MATLAB. The total cost PDFs corresponding to the stochastic control have a narrower shape compared to that of the deterministic control. This means that using the deterministic control, there is higher probability of getting both lower and higher cost values, hence more uncertainty. Moreover, the cost PDFs corresponding to the stochastic control are shifted to the left in figures 21, and 23. This shows that achieving lower total cost using the stochastic control is more probable. Since the multiplicative noise magnitude is proportional to the control effort, the control method limits the control magnitude to avoid incurring higher cost.

Through this numerical simulation, we have shown that the deterministic control is not able to compensate for the generated uncertainty. This in turn, demonstrates the robustness properties of the stochastic control in presence of disturbances.

6.3 *Stability of a Controlled Linear Stochastic System*

In the numerical experiments shown in this chapter, the linear deterministic control is seen to be able to regulate the stochastic system, since the noise magnitude is small. However, there may be scenarios depending on the physical and cost function parameters in which the linear deterministic control is not desirable. Consider a controlled linear SDE of the form:

$$dy_t = BKy_t dt + \varepsilon BKy_t dW_t, \quad y \in \mathbb{R}\,. \tag{6.4}$$

When the stochastic optimal control of section III is applied to the SDE (5.2), the coupling between x components in (5.2) is only through the nonlinear parts. Therefore, the scalar SDE (6.4) with K given by the optimal linear gain (5.45) is equivalent to one component in the linear part of the optimized (5.2). The process y_t given by (6.4) is a geometric Brownian motion,

$$y_t = y_0 e^{\left(BK - \frac{1}{2}\varepsilon^2 B^2 K^2\right)t + \varepsilon BKW_t},$$

where, without loss of generality, we assume that $y_0 > 0$. We also assume that y_0 is independent of W_t. Then, it can be determined that

$$\mathbb{E}[y_t] = \mathbb{E}[y_0]e^{BKt}, \qquad \mathbb{E}[y_t^2] = \mathbb{E}[y_0^2]e^{(2BK+\varepsilon^2 B^2 K^2)t}.$$

y_t is log-normally distributed with mean and variance $\mu_t = \mathbb{E}[y_t]$ and $\sigma_t^2 = \mathbb{E}[y_t^2] - \mu_t^2$, respectively.

Recall from (5.45) and (5.46) that the optimal linear stochastic gain K given by (5.18) is $K = -\frac{BP}{R+\varepsilon^2 B^2 P}$, where $R > 0$ is the weight of the control energy contribution to the cost function (5.1) and $P > 0$ is the solution to the ARE (5.19). Then, we see that y_t is stable in mean, because $BK < 0$, so $\mathbb{E}[y_t] \to 0$ as $t \to \infty$. y_t is also stable in mean-square sense, because

$$2BK + \varepsilon^2 B^2 K^2 = -(2R + \varepsilon^2 B^2 P) < 0,$$

hence $\mathbb{E}[y_t^2] \to 0$ as $t \to \infty$. In addition, for any $\delta > 0$,

$$\mathbb{P}[y_t > \delta] = \frac{1}{2}\left(1 - erf\left(\frac{\log \delta - \mu_t}{\sqrt{2}\sigma_t}\right)\right), \tag{6.5}$$

where erf is the error function. $\mu_t, \sigma_t \to 0$ as $t \to \infty$, so the error function in (6.5) goes to 1 as $t \to \infty$. Therefore, for any $\delta > 0$, $\mathbb{P}[y_t > \delta] \to 0$ as $t \to \infty$, i.e. y_t is stable in probability (the same conclusion can be made by an argument using the Markov inequality and $\mathbb{E}[y_t] \to 0$).

Now consider if we replaced K by the linear deterministic optimal gain $K_{det} = -\sqrt{\frac{Q}{R}}$. Then, the corresponding y_t will be stable in mean, but if

$$\frac{B}{2}\sqrt{\frac{Q}{R}} > \frac{1}{\varepsilon^2}, \tag{6.6}$$

then $(2BK_{det} + \varepsilon^2 B^2 K_{det}^2) > 0$, so y_t is no longer mean-square stable. Condition (6.6) indicates that if R is too small, then applying the deterministic optimal control may not ensure mean-square stability of the linear stochastic system. A small R means that we allow large control effort to be applied. Since noise enters the system proportionally to the control effort, it is not desirable to apply large control even if the cost function allows it. The stochastic optimal control accounts for this. The deterministic control however does not, and will try to apply large control which may destabilize the system in mean-square sense. Recall that B is inversely proportional to a principal moment of inertia. In theory, condition (6.6) could also be satisfied if a principal moment of inertia is too small.

The discussion on stability here is for a linear system, intended to investigate the possibility of scenarios in which applying a deterministic optimal control to a stochastic system is undesirable.

CHAPTER 7
Concluding Remarks

7.1 *Future Direction of Research*

Al'brekht Method, in context of attitude control, has left us several open problems to consider: One of which is the implementation of a path planning method that aims to reduce the optimality error. Al'brekht method gives the solution to the HJB locally around the origin. This means, that either the control, away from the origin, will lose its stability properties, or the optimality error will accumulate even when stable. Let us provide an overview of the HJB Residual planning method:

Suppose that we have solved the HJB equation associated with the deterministic controlled Euler rigid body dynamics given in section 2.1.

$$\dot{x} = f(x) + u(x), \text{ where } x \in \mathbb{R}^3, \text{ and } u \in \mathbb{R}^m.$$

Also, suppose that the associated HJB equation has been solved, and the value function up to some order, in form of a power series, is obtained. For example, assume that the we have up to the 5th order. Then, the value function partial sum is given as

$$\sum_{p=2}^{k} V^{(p)}(x), \qquad k \in \{2,3,4,5\} \tag{7.1}$$

Let us now define the HJB residual function

$$\mathcal{R}(x) = \frac{1}{2}\left[f(x)^T \frac{\partial V(x)}{\partial x} + \frac{1}{2}x^T Q x - \frac{1}{2}\left(\left(\frac{\partial V(x)}{\partial x}\right)^T BR^{-1}B^T \frac{\partial V(x)}{\partial x}\right)\right]^2 \tag{7.2}$$

Note that (7.2) is a known form in x. We then propose to use (7.2) as a potential function in a gradient descent method to generate an array of way-points P^i, $i = 1,2,3, \ldots$. The gradient descent equation is given by

$$p^{i+1} = p^i + \alpha^i \frac{\nabla_x \mathcal{R}(p^i)}{\|\mathcal{R}(p^i)\|} \tag{7.3}$$

where α^i are step size of the gradient. Then, at each way-point p^i, we will apply a coordinate shift to the Al'brekht's local control, such that the origin becomes the current way-point. This means that we will use each p^i as a shifted coordinate origin. Applying the control to reach each p^i successively, we will stabilize around each generated way-point. Note that the next way-point is not calculated until stabilization about the previous way point is achieved given a tolerance. This chain will be continued marching towards the global origin until the way-points reach the zero region of (7.2) . That is, when the gradient is not steep. This is the region where Al'brekht method will not generate significant optimality error since the HJB is close to zero, and hence satisfied.

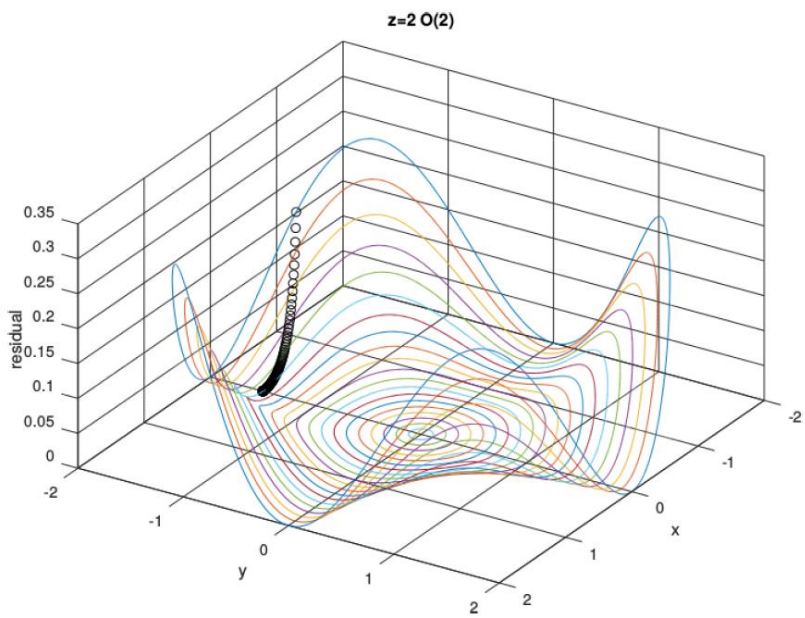

Figure 24. Contour of Residual for $V^{(2)}(x)$, Fixing $z = 2$

After this, Al'brekht control can be applied to the global origin. Though, planning can always resume, when the value exceeds a threshold.

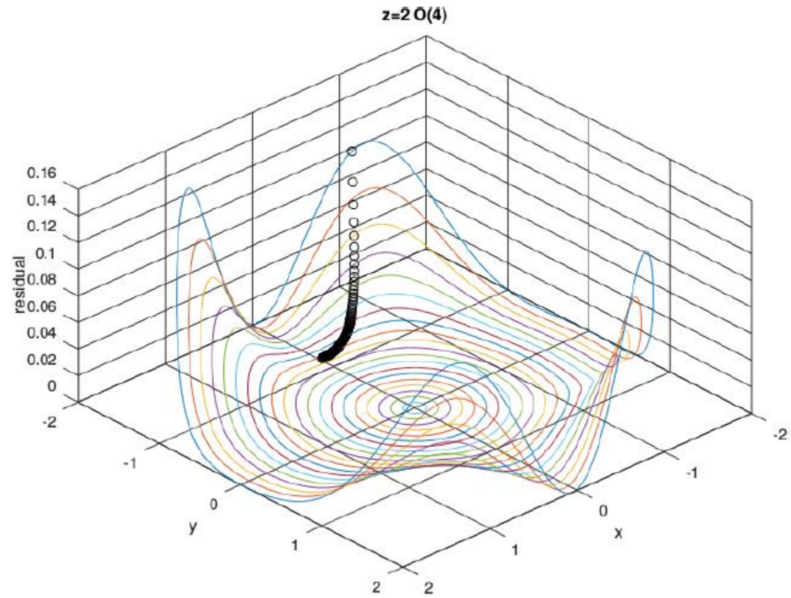

Figure 25. Contour of Residual for $V^{(4)}(x)$, Fixing $z = 2$

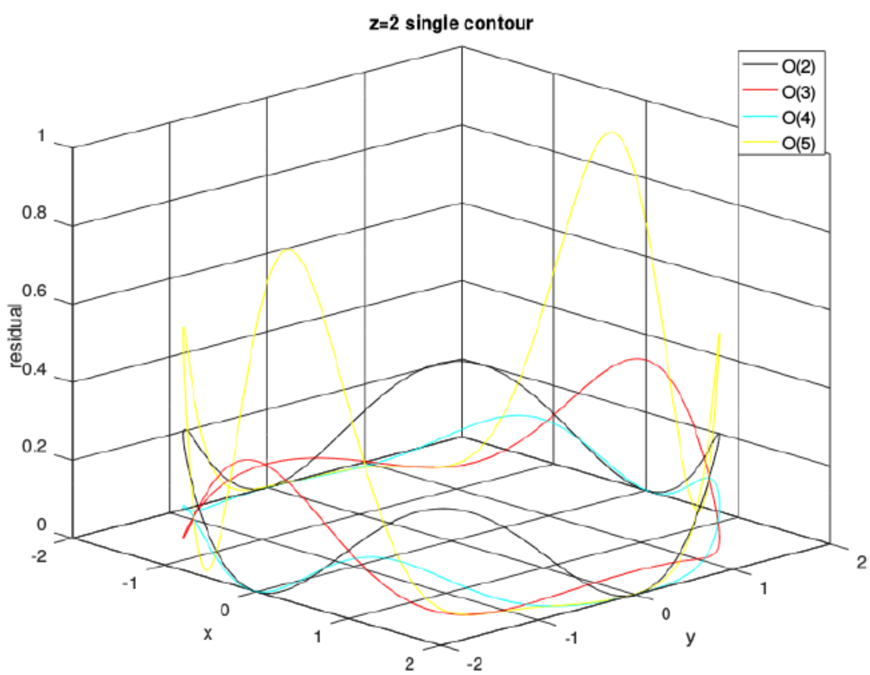

Figure 26. Residuals for $\sum_{p=2}^{k} V^{(p)}(x)$, $k \in \{2,3,4,5\}$, at Radius 2

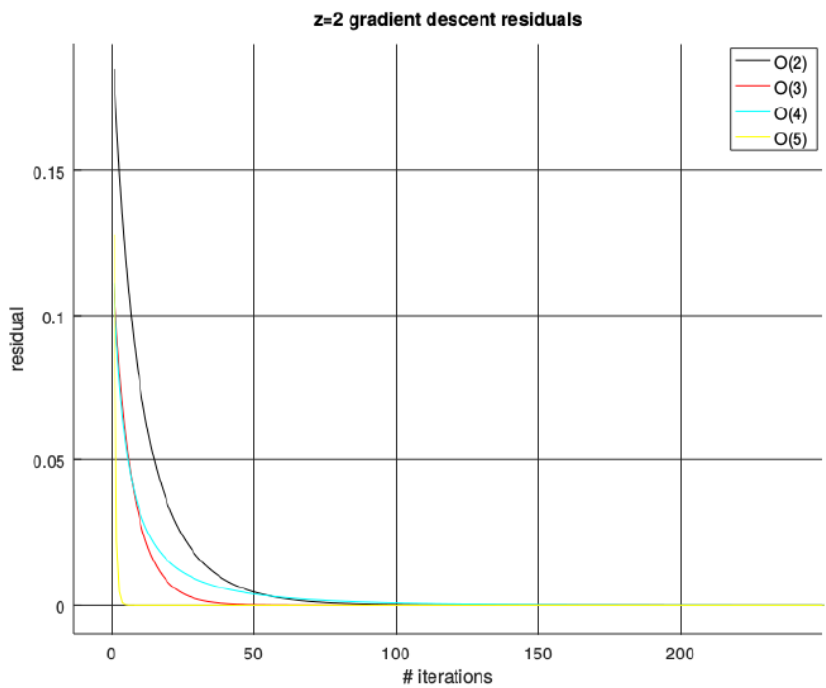

Figure 27. Residuals and Gradient Descent Iterations

This method is specifically useful because it improves the optimality results by finding the steepest descent direction on the manifold (7.2) and directing the Al'brekht control in the steepest direction to a region where it would generate a lower error, i.e. where HJB is satisfied with smaller error. This method gave an average of 30% improvement when the third order control with planning was compared to the third order Al'brekht control with no planning. However, computational challenges remain for future consideration: 1) Quantification of cost, or a proof of the method. 2) Determining the optimal choice of the time step α_i. In general, it is also desired to know if applying local solution to a series of shifted origins (i.e. the way-points) outside of the Al'brekht's region of convergence consecutively will stabilize the system, and in total yield lower optimality error.

7.2 *Conclusion*

Success of space missions depends on accuracy and efficiency of the spacecraft actuators. Thrust uncertainty causes state error, and accumulated state error is detrimental to mission objectives and can shorten the lifetime of the spacecraft. In fact, missions carrying science instruments and precision pointing devices will not be able to operate if thrust uncertainty grows. Motivated by these challenges, we began addressing the thrust uncertainty problem by modeling a realistic situation, where the generated thrust uncertainty is proportional to the magnitude of the generated thrust. To compensate for the thrust fluctuations, and reduce the fuel consumption under uncertainty, we formulated a stochastic optimal control problem through the Hamilton-Jacobi-Bellman equation. To solve the HJB equation, we extended Al'brekht [1] method for a stochastic setting. We then provided the stochastic solvability conditions both for quadratic and higher orders of the HJB solution. The HJB associated with the stochastic Euler dynamics was solved through the stochastic extension of the Al'brekht method [2]. Numerical experiments were carried out for a model of a 6U CubeSat. The optimality of the nonlinear stochastic control law was shown.

In this study, we were concerned with the stabilization of the attitude dynamics merely, and the attitude kinematics subsystem was ignored. As described earlier, for a given attitude parametrization, the dynamics and kinematics control laws can be jointly derived. That is, controlling both kinematics and dynamics at the same time. As a suggestion for future research, it would be important to understand the effects of thrust uncertainty on the actual pointing of the spacecraft. However, since the source of the noise is the dynamics model, it is important to study the control of noisy dynamics. Through analysis of detumbling maneuver of a 6U CubeSat model, we found that the stochastic control can minimize the optimality criteria 50% better than that of the linear deterministic control when uncertainty effects are large.

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APPENDIX A: Moment of Inertia Tables

Table 7. Table of Moment Inertia

Diagonal Tensor Entries	Off-Diagonal Tensor Entries
$I_{11} = \int \rho(r_2^2 + r_3^2)\, dV = \int (r_2^2 + r_3^2)\, dm$	$I_{12} = I_{21} = -\int \rho(r_1 r_2)\, dV = -\int (r_1 r_2)\, dm$
$I_{22} = \int \rho(r_1^2 + r_3^2)\, dV = \int (r_1^2 + r_3^2)\, dm$	$I_{13} = I_{31} = -\int \rho(r_1 r_3)\, dV = -\int (r_1 r_3)\, dm$
$I_{33} = \int \rho(r_1^2 + r_2^2)\, dV = \int (r_1^2 + r_2^2)\, dm$	$I_{23} = I_{32} = -\int \rho(r_2 r_3)\, dV = -\int (r_2 r_3)\, dm$

Table 8. Moment of Inertia of a Cuboid in Principal Axes

Diagonal Tensor Entries	Off-Diagonal Tensor Entries
$I_{11} = \frac{1}{12} m(x^2 + y^2)$	$I_{12} = I_{21} = 0$
$I_{22} = \frac{1}{12} m(z^2 + y^2)$	$I_{13} = I_{31} = 0$
$I_{33} = \frac{1}{12} m(x^2 + z^2)$	$I_{23} = I_{32} = 0$

where x is the width, y is the depth, z is the height, and m stands for the mass.

APPENDIX B: Monte Carlo Results in Regions II-VIII

This appendix contains the detailed results of the Monte Carlo experiment performed for a CubeSat model. The CubeSat parameters are given as: $I_{11} = 0.05$, $I_{22} = 0.065$, $I_{33} = 0.025$,

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, Q = R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For the results of Region I, reader may refer to chapter 6. There are two groups of Initial conditions: one group is randomly generated within distance of norm 1 of the origin, and the second group is randomly generated between norm 1 and norm 3. Each table contains 50 initials conditions (given in Table 2), where each initial condition has 2000 realizations. The displayed means are taken over 2000 realizations of each initial condition. Then average cost difference is calculated over all the 50 initial conditions and represents the percent improvement when a nonlinear stochastic is used.

Table 9. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region II and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.001787186	0.001756082	1.740407923
2	0.293187079	0.005038835	0.004920955	2.339434219
3	0.316221076	0.006312428	0.006088053	3.554507119
4	0.33126114	0.005102303	0.004663047	8.608970355
5	0.365447776	0.007284997	0.007379462	-1.296694577
6	0.388573818	0.008132459	0.007502579	7.745261206
7	0.414682564	0.009817469	0.009200311	6.286330293
8	0.507910302	0.013482402	0.013782249	-2.22399492
9	0.508723927	0.017571538	0.017437054	0.765350856
10	0.564035602	0.0208806	0.020954289	-0.352907078
11	0.568921476	0.020567418	0.019741404	4.016131555
12	0.621993608	0.01866271	0.015149173	18.82650885
13	0.622111707	0.025894273	0.025588377	1.181327011
14	0.652850918	0.023046459	0.023936338	-3.861238193
15	0.67303375	0.021150994	0.018784328	11.18938689

Table 9 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region II and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
16	0.674354319	0.029986569	0.029159949	2.75663266
17	0.683764431	0.021081104	0.020676911	1.917325655
18	0.684739975	0.022411294	0.022695257	-1.267049341
19	0.728289287	0.034520015	0.033816306	2.038552962
20	0.753863006	0.042627136	0.041004454	3.806687304
21	0.753969075	0.035957804	0.034113559	5.128914652
22	0.769293241	0.03291149	0.030016754	8.795520038
23	0.781665858	0.031843488	0.029949064	5.949174872
24	0.805212622	0.046007718	0.045669541	0.73504429
25	0.814479069	0.042407088	0.042819425	-0.972329771
26	0.817586623	0.0263043	0.023865614	9.271053816
27	0.820748706	0.044153573	0.04329998	1.933236148
28	0.833407694	0.0429302	0.043100726	-0.397216961
29	0.834535397	0.051409186	0.05125973	0.290718705
30	0.841453615	0.049813663	0.049421408	0.787444695
31	0.876241566	0.040243264	0.036707415	8.786188772
32	0.87893391	0.058238072	0.056162676	3.563640491
33	0.880603177	0.049914129	0.050197703	-0.568122256
34	0.881879127	0.031933613	0.032986491	-3.297084125
35	0.886511581	0.04568064	0.042883748	6.122707628
36	0.894100586	0.036652074	0.030782303	16.01484025
37	0.909989643	0.046310428	0.044307214	4.325621415
38	0.92952342	0.051793734	0.053668329	-3.619346401
39	0.933100196	0.053208718	0.052242295	1.8162855
40	0.933204146	0.054678481	0.052768732	3.492688046
41	0.942425394	0.054011861	0.051510826	4.630528721
42	0.964908062	0.043139319	0.039335993	8.8163807
43	0.967711414	0.062220081	0.061421123	1.284082947
44	0.975126961	0.070328078	0.072374025	-2.909147388
45	0.980126841	0.056292574	0.05209125	7.463372376
46	0.983925319	0.057858743	0.056570628	2.226310739
47	0.993427783	0.045255487	0.046352019	-2.422980985
48	0.993663086	0.05217928	0.048005282	7.999339039
49	0.994500751	0.063440051	0.062285013	1.820675837
50	0.998840156	0.06212588	0.060144351	3.18953923
			Average Cost Difference (%):	**3.360560235**

Table 10. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region II and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.070245779	0.068230392	2.869050437
52	1.329198658	0.117463472	0.109469492	6.805502619
53	1.652261099	0.143895357	0.139387729	3.132573433
54	1.732050808	0.179550232	0.173456861	3.393686053
55	1.760957841	0.214270745	0.211979152	1.069485073
56	1.820518877	0.217240747	0.209549624	3.540368458
57	1.828533372	0.229874129	0.223959981	2.572776917
58	1.893715218	0.204247576	0.191921918	6.034665388
59	1.900420695	0.219421845	0.216777136	1.205308088
60	1.969644102	0.207810879	0.187859942	9.600525839
61	1.976334082	0.233784174	0.2316903	0.89564389
62	2.002910229	0.228606799	0.230551107	-0.850503325
63	2.094697427	0.323991961	0.32081492	0.980592548
64	2.112206523	0.233710265	0.217858885	6.782492004
65	2.142249774	0.328948678	0.319311361	2.929732876
66	2.161830466	0.255456614	0.244001061	4.484344016
67	2.209623524	0.236789592	0.216612669	8.521034642
68	2.232543354	0.324032225	0.302356778	6.689287309
69	2.234850662	0.29042297	0.28514921	1.815889347
70	2.315624628	0.334264138	0.334716973	-0.135472379
71	2.386793676	0.407934943	0.414111166	-1.514021564
72	2.401746454	0.371028588	0.353334556	4.76891342
73	2.445361846	0.363890478	0.351063752	3.524886403
74	2.466706602	0.288768335	0.287397648	0.474666648
75	2.479790488	0.303431366	0.286379485	5.619683026
76	2.498843498	0.389911084	0.375446457	3.709724499
77	2.522092914	0.383458185	0.3879817	-1.179663181
78	2.551730261	0.41964923	0.385124659	8.227006941
79	2.620165462	0.343109175	0.281709368	17.89512256
80	2.649793028	0.390550632	0.394715022	-1.066286887
81	2.674902192	0.445465459	0.431014817	3.243942234
82	2.692003392	0.468729162	0.472361333	-0.774897567
83	2.701726462	0.49334873	0.447548344	9.283572398
84	2.714761477	0.501847977	0.497888644	0.788950623
85	2.749451218	0.456739879	0.451158763	1.221946398
86	2.795156227	0.492913987	0.485767194	1.449906688
87	2.808828434	0.469482248	0.468130921	0.28783342

Table 10 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region II and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	0.520561947	0.51202472	1.64000222
89	2.848903585	0.378694632	0.34827571	8.032572841
90	2.851611437	0.508560987	0.490061346	3.637644536
91	2.870713551	0.500184949	0.496479765	0.740762916
92	2.880350027	0.410602429	0.377497013	8.062645027
93	2.896534118	0.590504234	0.582117233	1.420311651
94	2.938218496	0.553349084	0.527290152	4.709311392
95	2.947634879	0.476814481	0.451724599	5.261979876
96	2.95110354	0.457190313	0.439490715	3.871385165
97	2.962064987	0.536774667	0.528172765	1.602516378
98	2.96452904	0.572060067	0.558076997	2.444335972
99	2.967810632	0.501318354	0.466710084	6.903451591
100	2.985712699	0.645811062	0.654568747	-1.356075395
			Average Cost Difference (%):	**3.505382269**

Table 11. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region II and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.003493445	0.003264176	6.562834423
2	0.293187079	0.015053933	0.008471753	43.72398541
3	0.316221076	0.013897618	0.010674692	23.1904957
4	0.33126114	0.025166066	0.009792621	61.08799577
5	0.365447776	0.018208594	0.012801079	29.69760086
6	0.388573818	0.026823411	0.01452616	45.84521754
7	0.414682564	0.044512341	0.018052068	59.44480142
8	0.507910302	0.192003712	0.03229315	83.18097639
9	0.508723927	0.044481629	0.028067132	36.90174597
10	0.564035602	0.13713477	0.03540153	74.18486217
11	0.568921476	0.089178261	0.037060238	58.44252028
12	0.621993608	0.714618753	0.044496448	93.77340051
13	0.622111707	0.056685316	0.050523537	10.87015069
14	0.652850918	0.079674009	0.041497823	47.91548315
15	0.67303375	0.115970855	0.059824317	48.41435235

Table 11 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region II and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.090249667	0.046152822	48.86095004
17	0.683764431	0.071015535	0.041598951	41.42274511
18	0.684739975	0.062165466	0.042550265	31.55321141
19	0.728289287	0.137318509	0.059229521	56.86705183
20	0.753863006	0.076281531	0.07292434	4.40105346
21	0.753969075	0.138699119	0.059190967	57.32419413
22	0.769293241	0.091591688	0.059692032	34.82811225
23	0.781665858	0.146891816	0.070907369	51.72816945
24	0.805212622	0.106633823	0.082595955	22.54244255
25	0.814479069	0.099691215	0.080274387	19.47697022
26	0.817586623	0.136042582	0.051932043	61.82662689
27	0.820748706	0.097524069	0.105040595	-7.707354635
28	0.833407694	0.123571258	0.069581791	43.69095823
29	0.834535397	0.096640046	0.08159809	15.56493082
30	0.841453615	0.101214707	0.089848657	11.22964221
31	0.876241566	0.132201754	0.090540221	31.51360075
32	0.87893391	0.117986956	0.084291491	28.55863577
33	0.880603177	0.110019419	0.09187209	16.49466059
34	0.881879127	0.170502414	0.266108195	-56.07297775
35	0.886511581	0.157848399	0.078608683	50.19988587
36	0.894100586	2.032361145	0.072069287	96.45391335
37	0.909989643	0.281443229	0.088091352	68.70013473
38	0.92952342	0.126805846	0.08245172	34.97798212
39	0.933100196	0.185981417	0.104590857	43.76273784
40	0.933204146	0.10741512	0.105385207	1.889783658
41	0.942425394	0.212769212	0.097594819	54.13113693
42	0.964908062	0.231812676	0.088514431	61.8163974
43	0.967711414	0.176188041	0.100265535	43.09174781
44	0.975126961	0.13192691	0.109216921	17.21406862
45	0.980126841	0.154903751	0.102251181	33.99050666
46	0.983925319	0.218861278	0.195686219	10.58892613
47	0.993427783	0.196843424	0.0989875	49.71256926
48	0.993663086	0.2024056	0.110881343	45.21824355
49	0.994500751	0.212804579	0.09939075	53.29482524
50	0.998840156	0.204202356	0.196492437	3.775627092
			Average Cost Difference (%):	**38.12317064**

Table 12. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region II and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.199268285	0.247438501	-24.17354875
52	1.329198658	1.642395547	0.207551646	87.36287089
53	1.652261099	0.563090231	0.271914883	51.71024674
54	1.732050808	0.457029415	0.318240024	30.36771516
55	1.760957841	0.53752857	0.339747365	36.79454751
56	1.820518877	0.589690053	0.434604814	26.29944983
57	1.828533372	0.549328134	0.355961554	35.20056017
58	1.893715218	0.801629778	0.349667461	56.38043011
59	1.900420695	0.760962932	0.372901251	50.99613453
60	1.969644102	0.619672077	0.383674592	38.08425358
61	1.976334082	0.610976003	0.417999149	31.58501357
62	2.002910229	0.558335626	0.428867605	23.18820705
63	2.094697427	0.715095973	0.508476819	28.8939054
64	2.112206523	0.697004219	0.585418469	16.00933636
65	2.142249774	1.671752979	0.52307355	68.71107412
66	2.161830466	2.339005584	0.508634741	78.25423143
67	2.209623524	2.913149444	0.422056995	85.5120033
68	2.232543354	1.073729951	0.608070775	43.36836988
69	2.234850662	0.848757895	0.600479439	29.25197603
70	2.315624628	1.239570132	0.586496239	52.68551381
71	2.386793676	1.879775141	0.675750158	64.05154302
72	2.401746454	1.215872837	0.628484547	48.31001006
73	2.445361846	1.962943323	0.697650601	64.4589534
74	2.466706602	0.83300508	0.540892673	35.06730199
75	2.479790488	0.930137753	0.62152214	33.17955985
76	2.498843498	1.199036	0.674847593	43.71748698
77	2.522092914	0.944717216	0.631720113	33.1313008
78	2.551730261	1.666905421	0.71847658	56.89757972
79	2.620165462	12.21159167	0.791960209	93.51468482
80	2.649793028	1.239420256	0.750476421	39.4493984
81	2.674902192	1.30986219	0.870943803	33.50874543
82	2.692003392	1.284168099	0.791930689	38.33122862
83	2.701726462	1.605677835	0.755005209	52.97903525
84	2.714761477	1.616452949	0.846707519	47.61941447
85	2.749451218	1.351313993	0.761567924	43.64241563
86	2.795156227	1.39371587	0.846905944	39.23395992
87	2.808828434	1.297656749	1.010098132	22.15983677

Table 12 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region II and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	1.525303493	0.835964752	45.19354637
89	2.848903585	1.610770268	0.727689055	54.82353567
90	2.851611437	1.22319458	0.916860317	25.04378844
91	2.870713551	1.588145517	0.90619685	42.93993589
92	2.880350027	3.114680512	0.810668377	73.97266352
93	2.896534118	1.288463281	1.300779489	-0.955883504
94	2.938218496	1.400484588	0.871475599	37.77328174
95	2.947634879	1.83657069	1.600178365	12.87139815
96	2.95110354	2.428432578	0.961442048	60.40894624
97	2.962064987	1.31687053	1.508836773	-14.57745758
98	2.96452904	2.622738637	0.920474722	64.90406217
99	2.967810632	1.274000862	0.876523724	31.19912634
100	2.985712699	1.340235132	0.96812078	27.76485586
			Average Cost Difference (%):	**41.9419309**

Table 13. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region III and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
1	0.157460926	0.001787186	0.001756082	1.740407923
2	0.293187079	0.005038835	0.004920955	2.339434219
3	0.316221076	0.006312428	0.006088053	3.554507119
4	0.33126114	0.005102303	0.004663047	8.608970355
5	0.365447776	0.007284997	0.007379462	-1.296694577
6	0.388573818	0.008132459	0.007502579	7.745261206
7	0.414682564	0.009817469	0.009200311	6.286330293
8	0.507910302	0.013482402	0.013782249	-2.22399492
9	0.508723927	0.017571538	0.017437054	0.765350856
10	0.564035602	0.0208806	0.020954289	-0.352907078
11	0.568921476	0.020567418	0.019741404	4.016131555
12	0.621993608	0.01866271	0.015149173	18.82650885
13	0.622111707	0.025894273	0.025588377	1.181327011
14	0.652850918	0.023046459	0.023936338	-3.861238193
15	0.67303375	0.021150994	0.018784328	11.18938689

Table 13 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region III and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.029986569	0.029159949	2.75663266
17	0.683764431	0.021081104	0.020676911	1.917325655
18	0.684739975	0.022411294	0.022695257	-1.267049341
19	0.728289287	0.034520015	0.033816306	2.038552962
20	0.753863006	0.042627136	0.041004454	3.806687304
21	0.753969075	0.035957804	0.034113559	5.128914652
22	0.769293241	0.03291149	0.030016754	8.795520038
23	0.781665858	0.031843488	0.029949064	5.949174872
24	0.805212622	0.046007718	0.045669541	0.73504429
25	0.814479069	0.042407088	0.042819425	-0.972329771
26	0.817586623	0.0263043	0.023865614	9.271053816
27	0.820748706	0.044153573	0.04329998	1.933236148
28	0.833407694	0.0429302	0.043100726	-0.397216961
29	0.834535397	0.051409186	0.05125973	0.290718705
30	0.841453615	0.049813663	0.049421408	0.787444695
31	0.876241566	0.040243264	0.036707415	8.786188772
32	0.87893391	0.058238072	0.056162676	3.563640491
33	0.880603177	0.049914129	0.050197703	-0.568122256
34	0.881879127	0.031933613	0.032986491	-3.297084125
35	0.886511581	0.04568064	0.042883748	6.122707628
36	0.894100586	0.036652074	0.030782303	16.01484025
37	0.909989643	0.046310428	0.044307214	4.325621415
38	0.92952342	0.051793734	0.053668329	-3.619346401
39	0.933100196	0.053208718	0.052242295	1.8162855
40	0.933204146	0.054678481	0.052768732	3.492688046
41	0.942425394	0.054011861	0.051510826	4.630528721
42	0.964908062	0.043139319	0.039335993	8.8163807
43	0.967711414	0.062220081	0.061421123	1.284082947
44	0.975126961	0.070328078	0.072374025	-2.909147388
45	0.980126841	0.056292574	0.05209125	7.463372376
46	0.983925319	0.057858743	0.056570628	2.226310739
47	0.993427783	0.045255487	0.046352019	-2.422980985
48	0.993663086	0.05217928	0.048005282	7.999339039
49	0.994500751	0.063440051	0.062285013	1.820675837
50	0.998840156	0.06212588	0.060144351	3.18953923
			Average Cost Difference (%):	**3.360560235**

Table 14. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region III and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.070245779	0.068230392	2.869050437
52	1.329198658	0.117463472	0.109469492	6.805502619
53	1.652261099	0.143895357	0.139387729	3.132573433
54	1.732050808	0.179550232	0.173456861	3.393686053
55	1.760957841	0.214270745	0.211979152	1.069485073
56	1.820518877	0.217240747	0.209549624	3.540368458
57	1.828533372	0.229874129	0.223959981	2.572776917
58	1.893715218	0.204247576	0.191921918	6.034665388
59	1.900420695	0.219421845	0.216777136	1.205308088
60	1.969644102	0.207810879	0.187859942	9.600525839
61	1.976334082	0.233784174	0.2316903	0.89564389
62	2.002910229	0.228606799	0.230551107	-0.850503325
63	2.094697427	0.323991961	0.32081492	0.980592548
64	2.112206523	0.233710265	0.217858885	6.782492004
65	2.142249774	0.328948678	0.319311361	2.929732876
66	2.161830466	0.255456614	0.244001061	4.484344016
67	2.209623524	0.236789592	0.216612669	8.521034642
68	2.232543354	0.324032225	0.302356778	6.689287309
69	2.234850662	0.29042297	0.28514921	1.815889347
70	2.315624628	0.334264138	0.334716973	-0.135472379
71	2.386793676	0.407934943	0.414111166	-1.514021564
72	2.401746454	0.371028588	0.353334556	4.76891342
73	2.445361846	0.363890478	0.351063752	3.524886403
74	2.466706602	0.288768335	0.287397648	0.474666648
75	2.479790488	0.303431366	0.286379485	5.619683026
76	2.498843498	0.389911084	0.375446457	3.709724499
77	2.522092914	0.383458185	0.3879817	-1.179663181
78	2.551730261	0.41964923	0.385124659	8.227006941
79	2.620165462	0.343109175	0.281709368	17.89512256
80	2.649793028	0.390550632	0.394715022	-1.066286887
81	2.674902192	0.445465459	0.431014817	3.243942234
82	2.692003392	0.468729162	0.472361333	-0.774897567
83	2.701726462	0.49334873	0.447548344	9.283572398
84	2.714761477	0.501847977	0.497888644	0.788950623
85	2.749451218	0.456739879	0.451158763	1.221946398
86	2.795156227	0.492913987	0.485767194	1.449906688
87	2.808828434	0.469482248	0.468130921	0.28783342

Table 14 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region III and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	0.520561947	0.51202472	1.64000222
89	2.848903585	0.378694632	0.34827571	8.032572841
90	2.851611437	0.508560987	0.490061346	3.637644536
91	2.870713551	0.500184949	0.496479765	0.740762916
92	2.880350027	0.410602429	0.377497013	8.062645027
93	2.896534118	0.590504234	0.582117233	1.420311651
94	2.938218496	0.553349084	0.527290152	4.709311392
95	2.947634879	0.476814481	0.451724599	5.261979876
96	2.95110354	0.457190313	0.439490715	3.871385165
97	2.962064987	0.536774667	0.528172765	1.602516378
98	2.96452904	0.572060067	0.558076997	2.444335972
99	2.967810632	0.501318354	0.466710084	6.903451591
100	2.985712699	0.645811062	0.654568747	-1.356075395
			Average Cost Difference (%):	**3.505382269**

Table 15. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region III and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
1	0.157460926	0.003493445	0.003264176	6.562834423
2	0.293187079	0.015053933	0.008471753	43.72398541
3	0.316221076	0.013897618	0.010674692	23.1904957
4	0.33126114	0.025166066	0.009792621	61.08799577
5	0.365447776	0.018208594	0.012801079	29.69760086
6	0.388573818	0.026823411	0.01452616	45.84521754
7	0.414682564	0.044512341	0.018052068	59.44480142
8	0.507910302	0.192003712	0.03229315	83.18097639
9	0.508723927	0.044481629	0.028067132	36.90174597
10	0.564035602	0.13713477	0.03540153	74.18486217
11	0.568921476	0.089178261	0.037060238	58.44252028
12	0.621993608	0.714618753	0.044496448	93.77340051
13	0.622111707	0.056685316	0.050523537	10.87015069
14	0.652850918	0.079674009	0.041497823	47.91548315
15	0.67303375	0.115970855	0.059824317	48.41435235

Table 15 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region III and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.090249667	0.046152822	48.86095004
17	0.683764431	0.071015535	0.041598951	41.42274511
18	0.684739975	0.062165466	0.042550265	31.55321141
19	0.728289287	0.137318509	0.059229521	56.86705183
20	0.753863006	0.076281531	0.07292434	4.40105346
21	0.753969075	0.138699119	0.059190967	57.32419413
22	0.769293241	0.091591688	0.059692032	34.82811225
23	0.781665858	0.146891816	0.070907369	51.72816945
24	0.805212622	0.106633823	0.082595955	22.54244255
25	0.814479069	0.099691215	0.080274387	19.47697022
26	0.817586623	0.136042582	0.051932043	61.82662689
27	0.820748706	0.097524069	0.105040595	-7.707354635
28	0.833407694	0.123571258	0.069581791	43.69095823
29	0.834535397	0.096640046	0.08159809	15.56493082
30	0.841453615	0.101214707	0.089848657	11.22964221
31	0.876241566	0.132201754	0.090540221	31.51360075
32	0.87893391	0.117986956	0.084291491	28.55863577
33	0.880603177	0.110019419	0.09187209	16.49466059
34	0.881879127	0.170502414	0.266108195	-56.07297775
35	0.886511581	0.157848399	0.078608683	50.19988587
36	0.894100586	2.032361145	0.072069287	96.45391335
37	0.909989643	0.281443229	0.088091352	68.70013473
38	0.92952342	0.126805846	0.08245172	34.97798212
39	0.933100196	0.185981417	0.104590857	43.76273784
40	0.933204146	0.10741512	0.105385207	1.889783658
41	0.942425394	0.212769212	0.097594819	54.13113693
42	0.964908062	0.231812676	0.088514431	61.8163974
43	0.967711414	0.176188041	0.100265535	43.09174781
44	0.975126961	0.13192691	0.109216921	17.21406862
45	0.980126841	0.154903751	0.102251181	33.99050666
46	0.983925319	0.218861278	0.195686219	10.58892613
47	0.993427783	0.196843424	0.0989875	49.71256926
48	0.993663086	0.2024056	0.110881343	45.21824355
49	0.994500751	0.212804579	0.09939075	53.29482524
50	0.998840156	0.204202356	0.196492437	3.775627092
			Average Cost Difference (%):	**38.12317064**

Table 16. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region III and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.199268285	0.247438501	-24.17354875
52	1.329198658	1.642395547	0.207551646	87.36287089
53	1.652261099	0.563090231	0.271914883	51.71024674
54	1.732050808	0.457029415	0.318240024	30.36771516
55	1.760957841	0.53752857	0.339747365	36.79454751
56	1.820518877	0.589690053	0.434604814	26.29944983
57	1.828533372	0.549328134	0.355961554	35.20056017
58	1.893715218	0.801629778	0.349667461	56.38043011
59	1.900420695	0.760962932	0.372901251	50.99613453
60	1.969644102	0.619672077	0.383674592	38.08425358
61	1.976334082	0.610976003	0.417999149	31.58501357
62	2.002910229	0.558335626	0.428867605	23.18820705
63	2.094697427	0.715095973	0.508476819	28.8939054
64	2.112206523	0.697004219	0.585418469	16.00933636
65	2.142249774	1.671752979	0.52307355	68.71107412
66	2.161830466	2.339005584	0.508634741	78.25423143
67	2.209623524	2.913149444	0.422056995	85.5120033
68	2.232543354	1.073729951	0.608070775	43.36836988
69	2.234850662	0.848757895	0.600479439	29.25197603
70	2.315624628	1.239570132	0.586496239	52.68551381
71	2.386793676	1.879775141	0.675750158	64.05154302
72	2.401746454	1.215872837	0.628484547	48.31001006
73	2.445361846	1.962943323	0.697650601	64.4589534
74	2.466706602	0.83300508	0.540892673	35.06730199
75	2.479790488	0.930137753	0.62152214	33.17955985
76	2.498843498	1.199036	0.674847593	43.71748698
77	2.522092914	0.944717216	0.631720113	33.1313008
78	2.551730261	1.666905421	0.71847658	56.89757972
79	2.620165462	12.21159167	0.791960209	93.51468482
80	2.649793028	1.239420256	0.750476421	39.4493984
81	2.674902192	1.30986219	0.870943803	33.50874543
82	2.692003392	1.284168099	0.791930689	38.33122862
83	2.701726462	1.605677835	0.755005209	52.97903525
84	2.714761477	1.616452949	0.846707519	47.61941447
85	2.749451218	1.351313993	0.761567924	43.64241563
86	2.795156227	1.39371587	0.846905944	39.23395992
87	2.808828434	1.297656749	1.010098132	22.15983677

Table 16 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region III and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	1.525303493	0.835964752	45.19354637
89	2.848903585	1.610770268	0.727689055	54.82353567
90	2.851611437	1.22319458	0.916860317	25.04378844
91	2.870713551	1.588145517	0.90619685	42.93993589
92	2.880350027	3.114680512	0.810668377	73.97266352
93	2.896534118	1.288463281	1.300779489	-0.955883504
94	2.938218496	1.400484588	0.871475599	37.77328174
95	2.947634879	1.83657069	1.600178365	12.87139815
96	2.95110354	2.428432578	0.961442048	60.40894624
97	2.962064987	1.31687053	1.508836773	-14.57745758
98	2.96452904	2.622738637	0.920474722	64.90406217
99	2.967810632	1.274000862	0.876523724	31.19912634
100	2.985712699	1.340235132	0.96812078	27.76485586
			Average Cost Difference (%):	**41.9419309**

Table 17. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region IV and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.001787186	0.001756082	1.740407923
2	0.293187079	0.005038835	0.004920955	2.339434219
3	0.316221076	0.006312428	0.006088053	3.554507119
4	0.33126114	0.005102303	0.004663047	8.608970355
5	0.365447776	0.007284997	0.007379462	-1.296694577
6	0.388573818	0.008132459	0.007502579	7.745261206
7	0.414682564	0.009817469	0.009200311	6.286330293
8	0.507910302	0.013482402	0.013782249	-2.22399492
9	0.508723927	0.017571538	0.017437054	0.765350856
10	0.564035602	0.0208806	0.020954289	-0.352907078
11	0.568921476	0.020567418	0.019741404	4.016131555
12	0.621993608	0.01866271	0.015149173	18.82650885
13	0.622111707	0.025894273	0.025588377	1.181327011
14	0.652850918	0.023046459	0.023936338	-3.861238193
15	0.67303375	0.021150994	0.018784328	11.18938689

Table 17 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region IV and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.029986569	0.029159949	2.75663266
17	0.683764431	0.021081104	0.020676911	1.917325655
18	0.684739975	0.022411294	0.022695257	-1.267049341
19	0.728289287	0.034520015	0.033816306	2.038552962
20	0.753863006	0.042627136	0.041004454	3.806687304
21	0.753969075	0.035957804	0.034113559	5.128914652
22	0.769293241	0.03291149	0.030016754	8.795520038
23	0.781665858	0.031843488	0.029949064	5.949174872
24	0.805212622	0.046007718	0.045669541	0.73504429
25	0.814479069	0.042407088	0.042819425	-0.972329771
26	0.817586623	0.0263043	0.023865614	9.271053816
27	0.820748706	0.044153573	0.04329998	1.933236148
28	0.833407694	0.0429302	0.043100726	-0.397216961
29	0.834535397	0.051409186	0.05125973	0.290718705
30	0.841453615	0.049813663	0.049421408	0.787444695
31	0.876241566	0.040243264	0.036707415	8.786188772
32	0.87893391	0.058238072	0.056162676	3.563640491
33	0.880603177	0.049914129	0.050197703	-0.568122256
34	0.881879127	0.031933613	0.032986491	-3.297084125
35	0.886511581	0.04568064	0.042883748	6.122707628
36	0.894100586	0.036652074	0.030782303	16.01484025
37	0.909989643	0.046310428	0.044307214	4.325621415
38	0.92952342	0.051793734	0.053668329	-3.619346401
39	0.933100196	0.053208718	0.052242295	1.8162855
40	0.933204146	0.054678481	0.052768732	3.492688046
41	0.942425394	0.054011861	0.051510826	4.630528721
42	0.964908062	0.043139319	0.039335993	8.8163807
43	0.967711414	0.062220081	0.061421123	1.284082947
44	0.975126961	0.070328078	0.072374025	-2.909147388
45	0.980126841	0.056292574	0.05209125	7.463372376
46	0.983925319	0.057858743	0.056570628	2.226310739
47	0.993427783	0.045255487	0.046352019	-2.422980985
48	0.993663086	0.05217928	0.048005282	7.999339039
49	0.994500751	0.063440051	0.062285013	1.820675837
50	0.998840156	0.06212588	0.060144351	3.18953923
			Average Cost Difference (%):	**3.360560235**

Table 18. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region IV and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.070245779	0.068230392	2.869050437
52	1.329198658	0.117463472	0.109469492	6.805502619
53	1.652261099	0.143895357	0.139387729	3.132573433
54	1.732050808	0.179550232	0.173456861	3.393686053
55	1.760957841	0.214270745	0.211979152	1.069485073
56	1.820518877	0.217240747	0.209549624	3.540368458
57	1.828533372	0.229874129	0.223959981	2.572776917
58	1.893715218	0.204247576	0.191921918	6.034665388
59	1.900420695	0.219421845	0.216777136	1.205308088
60	1.969644102	0.207810879	0.187859942	9.600525839
61	1.976334082	0.233784174	0.2316903	0.89564389
62	2.002910229	0.228606799	0.230551107	-0.850503325
63	2.094697427	0.323991961	0.32081492	0.980592548
64	2.112206523	0.233710265	0.217858885	6.782492004
65	2.142249774	0.328948678	0.319311361	2.929732876
66	2.161830466	0.255456614	0.244001061	4.484344016
67	2.209623524	0.236789592	0.216612669	8.521034642
68	2.232543354	0.324032225	0.302356778	6.689287309
69	2.234850662	0.29042297	0.28514921	1.815889347
70	2.315624628	0.334264138	0.334716973	-0.135472379
71	2.386793676	0.407934943	0.414111166	-1.514021564
72	2.401746454	0.371028588	0.353334556	4.76891342
73	2.445361846	0.363890478	0.351063752	3.524886403
74	2.466706602	0.288768335	0.287397648	0.474666648
75	2.479790488	0.303431366	0.286379485	5.619683026
76	2.498843498	0.389911084	0.375446457	3.709724499
77	2.522092914	0.383458185	0.3879817	-1.179663181
78	2.551730261	0.41964923	0.385124659	8.227006941
79	2.620165462	0.343109175	0.281709368	17.89512256
80	2.649793028	0.390550632	0.394715022	-1.066286887
81	2.674902192	0.445465459	0.431014817	3.243942234
82	2.692003392	0.468729162	0.472361333	-0.774897567
83	2.701726462	0.49334873	0.447548344	9.283572398
84	2.714761477	0.501847977	0.497888644	0.788950623
85	2.749451218	0.456739879	0.451158763	1.221946398
86	2.795156227	0.492913987	0.485767194	1.449906688
87	2.808828434	0.469482248	0.468130921	0.28783342

Table 18 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region IV and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	0.520561947	0.51202472	1.64000222
89	2.848903585	0.378694632	0.34827571	8.032572841
90	2.851611437	0.508560987	0.490061346	3.637644536
91	2.870713551	0.500184949	0.496479765	0.740762916
92	2.880350027	0.410602429	0.377497013	8.062645027
93	2.896534118	0.590504234	0.582117233	1.420311651
94	2.938218496	0.553349084	0.527290152	4.709311392
95	2.947634879	0.476814481	0.451724599	5.261979876
96	2.95110354	0.457190313	0.439490715	3.871385165
97	2.962064987	0.536774667	0.528172765	1.602516378
98	2.96452904	0.572060067	0.558076997	2.444335972
99	2.967810632	0.501318354	0.466710084	6.903451591
100	2.985712699	0.645811062	0.654568747	-1.356075395
			Average Cost Difference (%):	**3.505382269**

Table 19. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region IV and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.003493445	0.003264176	6.562834423
2	0.293187079	0.015053933	0.008471753	43.72398541
3	0.316221076	0.013897618	0.010674692	23.1904957
4	0.33126114	0.025166066	0.009792621	61.08799577
5	0.365447776	0.018208594	0.012801079	29.69760086
6	0.388573818	0.026823411	0.01452616	45.84521754
7	0.414682564	0.044512341	0.018052068	59.44480142
8	0.507910302	0.192003712	0.03229315	83.18097639
9	0.508723927	0.044481629	0.028067132	36.90174597
10	0.564035602	0.13713477	0.03540153	74.18486217
11	0.568921476	0.089178261	0.037060238	58.44252028
12	0.621993608	0.714618753	0.044496448	93.77340051
13	0.622111707	0.056685316	0.050523537	10.87015069
14	0.652850918	0.079674009	0.041497823	47.91548315
15	0.67303375	0.115970855	0.059824317	48.41435235

Table 19 (cont.). Mean Cost Comparison of the Linear Deterministic and the Nonlinear Stochastic Control with $\varepsilon = 0.28$ in Region IV and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.090249667	0.046152822	48.86095004
17	0.683764431	0.071015535	0.041598951	41.42274511
18	0.684739975	0.062165466	0.042550265	31.55321141
19	0.728289287	0.137318509	0.059229521	56.86705183
20	0.753863006	0.076281531	0.07292434	4.40105346
21	0.753969075	0.138699119	0.059190967	57.32419413
22	0.769293241	0.091591688	0.059692032	34.82811225
23	0.781665858	0.146891816	0.070907369	51.72816945
24	0.805212622	0.106633823	0.082595955	22.54244255
25	0.814479069	0.099691215	0.080274387	19.47697022
26	0.817586623	0.136042582	0.051932043	61.82662689
27	0.820748706	0.097524069	0.105040595	-7.707354635
28	0.833407694	0.123571258	0.069581791	43.69095823
29	0.834535397	0.096640046	0.08159809	15.56493082
30	0.841453615	0.101214707	0.089848657	11.22964221
31	0.876241566	0.132201754	0.090540221	31.51360075
32	0.87893391	0.117986956	0.084291491	28.55863577
33	0.880603177	0.110019419	0.09187209	16.49466059
34	0.881879127	0.170502414	0.266108195	-56.07297775
35	0.886511581	0.157848399	0.078608683	50.19988587
36	0.894100586	2.032361145	0.072069287	96.45391335
37	0.909989643	0.281443229	0.088091352	68.70013473
38	0.92952342	0.126805846	0.08245172	34.97798212
39	0.933100196	0.185981417	0.104590857	43.76273784
40	0.933204146	0.10741512	0.105385207	1.889783658
41	0.942425394	0.212769212	0.097594819	54.13113693
42	0.964908062	0.231812676	0.088514431	61.8163974
43	0.967711414	0.176188041	0.100265535	43.09174781
44	0.975126961	0.13192691	0.109216921	17.21406862
45	0.980126841	0.154903751	0.102251181	33.99050666
46	0.983925319	0.218861278	0.195686219	10.58892613
47	0.993427783	0.196843424	0.0989875	49.71256926
48	0.993663086	0.2024056	0.110881343	45.21824355
49	0.994500751	0.212804579	0.09939075	53.29482524
50	0.998840156	0.204202356	0.196492437	3.775627092
			Average Cost Difference (%):	**38.12317064**

Table 20. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region IV and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
51	1.168965631	0.199268285	0.247438501	-24.17354875
52	1.329198658	1.642395547	0.207551646	87.36287089
53	1.652261099	0.563090231	0.271914883	51.71024674
54	1.732050808	0.457029415	0.318240024	30.36771516
55	1.760957841	0.53752857	0.339747365	36.79454751
56	1.820518877	0.589690053	0.434604814	26.29944983
57	1.828533372	0.549328134	0.355961554	35.20056017
58	1.893715218	0.801629778	0.349667461	56.38043011
59	1.900420695	0.760962932	0.372901251	50.99613453
60	1.969644102	0.619672077	0.383674592	38.08425358
61	1.976334082	0.610976003	0.417999149	31.58501357
62	2.002910229	0.558335626	0.428867605	23.18820705
63	2.094697427	0.715095973	0.508476819	28.8939054
64	2.112206523	0.697004219	0.585418469	16.00933636
65	2.142249774	1.671752979	0.52307355	68.71107412
66	2.161830466	2.339005584	0.508634741	78.25423143
67	2.209623524	2.913149444	0.422056995	85.5120033
68	2.232543354	1.073729951	0.608070775	43.36836988
69	2.234850662	0.848757895	0.600479439	29.25197603
70	2.315624628	1.239570132	0.586496239	52.68551381
71	2.386793676	1.879775141	0.675750158	64.05154302
72	2.401746454	1.215872837	0.628484547	48.31001006
73	2.445361846	1.962943323	0.697650601	64.4589534
74	2.466706602	0.83300508	0.540892673	35.06730199
75	2.479790488	0.930137753	0.62152214	33.17955985
76	2.498843498	1.199036	0.674847593	43.71748698
77	2.522092914	0.944717216	0.631720113	33.1313008
78	2.551730261	1.666905421	0.71847658	56.89757972
79	2.620165462	12.21159167	0.791960209	93.51468482
80	2.649793028	1.239420256	0.750476421	39.4493984
81	2.674902192	1.30986219	0.870943803	33.50874543
82	2.692003392	1.284168099	0.791930689	38.33122862
83	2.701726462	1.605677835	0.755005209	52.97903525
84	2.714761477	1.616452949	0.846707519	47.61941447
85	2.749451218	1.351313993	0.761567924	43.64241563
86	2.795156227	1.39371587	0.846905944	39.23395992
87	2.808828434	1.297656749	1.010098132	22.15983677

Table 20 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region IV and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	1.525303493	0.835964752	45.19354637
89	2.848903585	1.610770268	0.727689055	54.82353567
90	2.851611437	1.22319458	0.916860317	25.04378844
91	2.870713551	1.588145517	0.90619685	42.93993589
92	2.880350027	3.114680512	0.810668377	73.97266352
93	2.896534118	1.288463281	1.300779489	-0.955883504
94	2.938218496	1.400484588	0.871475599	37.77328174
95	2.947634879	1.83657069	1.600178365	12.87139815
96	2.95110354	2.428432578	0.961442048	60.40894624
97	2.962064987	1.31687053	1.508836773	-14.57745758
98	2.96452904	2.622738637	0.920474722	64.90406217
99	2.967810632	1.274000862	0.876523724	31.19912634
100	2.985712699	1.340235132	0.96812078	27.76485586
			Average Cost Difference (%):	**41.9419309**

Table 21. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region V and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.001787112	0.001756035	1.738940351
2	0.293187079	0.005037878	0.004920388	2.33213718
3	0.316221076	0.006311412	0.0060878	3.542977192
4	0.33126114	0.005101999	0.004662632	8.611661092
5	0.365447776	0.007284514	0.00737724	-1.272922766
6	0.388573818	0.008131774	0.00750104	7.756422708
7	0.414682564	0.009815822	0.009198391	6.290166693
8	0.507910302	0.013482154	0.013781099	-2.217341664
9	0.508723927	0.017571308	0.01743703	0.764188787
10	0.564035602	0.020880404	0.020954155	-0.353204254
11	0.568921476	0.020557525	0.019734541	4.003320784
12	0.621993608	0.01865852	0.015146834	18.82081288
13	0.622111707	0.025891535	0.025585309	1.182726688
14	0.652850918	0.023042072	0.023932802	-3.865666935
15	0.67303375	0.021142996	0.018779783	11.17728404

Table 21 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region V and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.029970584	0.029152984	2.728006034
17	0.683764431	0.021077498	0.020672257	1.922623535
18	0.684739975	0.022408794	0.022692774	-1.267268872
19	0.728289287	0.034512863	0.033811517	2.032128611
20	0.753863006	0.04262121	0.040998754	3.806687817
21	0.753969075	0.035947219	0.034101678	5.134030623
22	0.769293241	0.032891125	0.0299939	8.808531349
23	0.781665858	0.03182285	0.029936885	5.926451621
24	0.805212622	0.046005618	0.045666789	0.736494942
25	0.814479069	0.042395205	0.042805415	-0.967585043
26	0.817586623	0.02630429	0.023865562	9.271215112
27	0.820748706	0.044133785	0.043270799	1.955386346
28	0.833407694	0.042913722	0.043080878	-0.3895156
29	0.834535397	0.051402634	0.051255261	0.286704836
30	0.841453615	0.049812823	0.049420447	0.787700048
31	0.876241566	0.040240983	0.036699145	8.801569929
32	0.87893391	0.058238051	0.05616268	3.563599079
33	0.880603177	0.04989527	0.05018582	-0.582320692
34	0.881879127	0.031929159	0.032982955	-3.300417616
35	0.886511581	0.045672404	0.042879506	6.115067977
36	0.894100586	0.036650603	0.030782172	16.01182656
37	0.909989643	0.046288952	0.044261995	4.378923414
38	0.92952342	0.051760801	0.053652673	-3.655027316
39	0.933100196	0.053189866	0.052229509	1.805525512
40	0.933204146	0.054676087	0.052767209	3.491248693
41	0.942425394	0.053969841	0.051481763	4.610126959
42	0.964908062	0.043131534	0.039325133	8.825099542
43	0.967711414	0.062165861	0.061383382	1.258694961
44	0.975126961	0.070327167	0.072373606	-2.909884475
45	0.980126841	0.056259329	0.052075052	7.437480215
46	0.983925319	0.057841676	0.056535269	2.258591583
47	0.993427783	0.045246154	0.046325056	-2.384517571
48	0.993663086	0.052165783	0.047989905	8.005013569
49	0.994500751	0.063419471	0.062259916	1.828390743
50	0.998840156	0.062108323	0.060126635	3.190696826
			Average Cost Difference (%):	**3.36065564**

Table 22. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region V and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
51	1.168965631	0.070191713	0.068150652	2.9078377
52	1.329198658	0.117361029	0.109460345	6.731948643
53	1.652261099	0.143674911	0.139235204	3.090106102
54	1.732050808	0.179303516	0.173288531	3.354638773
55	1.760957841	0.214096262	0.211875683	1.037187152
56	1.820518877	0.217084647	0.209392151	3.543546722
57	1.828533372	0.229761775	0.223829047	2.582121236
58	1.893715218	0.204129525	0.191760085	6.059603656
59	1.900420695	0.219215456	0.216525443	1.227109244
60	1.969644102	0.20751687	0.187530226	9.631334716
61	1.976334082	0.233608943	0.231516263	0.895804623
62	2.002910229	0.228263976	0.230335222	-0.907390959
63	2.094697427	0.323792476	0.320701762	0.954535557
64	2.112206523	0.233609617	0.217494183	6.898446109
65	2.142249774	0.328475928	0.319032331	2.87497391
66	2.161830466	0.255047446	0.243609678	4.484564972
67	2.209623524	0.23657631	0.216357605	8.546377989
68	2.232543354	0.323913437	0.302221315	6.696888779
69	2.234850662	0.290089866	0.28477596	1.831813683
70	2.315624628	0.333838442	0.334374034	-0.160434653
71	2.386793676	0.407853192	0.413731914	-1.441381919
72	2.401746454	0.370879516	0.35299893	4.821130554
73	2.445361846	0.363535147	0.350316715	3.636080996
74	2.466706602	0.288637436	0.286798041	0.637268558
75	2.479790488	0.303081871	0.285979794	5.642725251
76	2.498843498	0.389289977	0.375051423	3.657570315
77	2.522092914	0.382799393	0.387668618	-1.272004341
78	2.551730261	0.419035057	0.385241011	8.064730043
79	2.620165462	0.342699584	0.281481046	17.86361591
80	2.649793028	0.390258268	0.393872377	-0.926081435
81	2.674902192	0.445186068	0.430648386	3.265529415
82	2.692003392	0.468286689	0.471840375	-0.758869871
83	2.701726462	0.491347485	0.446572784	9.112634549
84	2.714761477	0.501169848	0.497433609	0.745503644
85	2.749451218	0.455737995	0.450722332	1.10055848
86	2.795156227	0.49244978	0.485167592	1.478767706
87	2.808828434	0.468940135	0.467790329	0.245192429

Table 22 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region V and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	0.519508891	0.511291074	1.581843384
89	2.848903585	0.378647326	0.348047163	8.081441586
90	2.851611437	0.507423167	0.489537912	3.524721758
91	2.870713551	0.500022338	0.496462186	0.711998444
92	2.880350027	0.40937772	0.376648086	7.994971868
93	2.896534118	0.589656375	0.58144814	1.392036967
94	2.938218496	0.55303249	0.527127318	4.684204344
95	2.947634879	0.476586988	0.451252678	5.315778801
96	2.95110354	0.456732073	0.439179763	3.843021138
97	2.962064987	0.53606848	0.527688006	1.56332165
98	2.96452904	0.571488323	0.557432346	2.459538829
99	2.967810632	0.499924199	0.466361881	6.713481384
100	2.985712699	0.645541065	0.654191947	-1.340097886
			Average Cost Difference (%):	**3.49360493**

Table 23. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region V and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.00349229	0.003264837	6.513001516
2	0.293187079	0.015058367	0.008467066	43.77168287
3	0.316221076	0.013889551	0.010672955	23.15838909
4	0.33126114	0.025155954	0.009789764	61.08371046
5	0.365447776	0.018187926	0.012794015	29.65655186
6	0.388573818	0.026858774	0.014510821	45.97362831
7	0.414682564	0.044570523	0.018050496	59.50126989
8	0.507910302	0.191886474	0.032275764	83.17976101
9	0.508723927	0.044473255	0.028064927	36.8948222
10	0.564035602	0.13713821	0.035395664	74.18978703
11	0.568921476	0.08890041	0.037000485	58.37984876
12	0.621993608	0.714279515	0.04448058	93.77266475
13	0.622111707	0.056667494	0.050507944	10.86963571
14	0.652850918	0.07955313	0.041483741	47.85404237
15	0.67303375	0.116051631	0.059718463	48.54146981

Table 23 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region V and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
16	0.674354319	0.089801276	0.046095066	48.66992116
17	0.683764431	0.070920477	0.041553911	41.40773901
18	0.684739975	0.061879376	0.042537097	31.25803803
19	0.728289287	0.137090686	0.059161045	56.84532147
20	0.753863006	0.076017819	0.07290567	4.093973022
21	0.753969075	0.135914651	0.059152943	56.47787627
22	0.769293241	0.091202424	0.059648957	34.5971803
23	0.781665858	0.146675045	0.070940608	51.63416638
24	0.805212622	0.106568052	0.082599594	22.49122271
25	0.814479069	0.099776774	0.081304137	18.51396426
26	0.817586623	0.136041409	0.051932838	61.82571286
27	0.820748706	0.097665928	0.104537157	-7.035441409
28	0.833407694	0.122711451	0.069471373	43.38639741
29	0.834535397	0.096651786	0.081547995	15.62701662
30	0.841453615	0.101273018	0.089838269	11.29101214
31	0.876241566	0.132275	0.090425568	31.63820246
32	0.87893391	0.117986472	0.084291419	28.55840345
33	0.880603177	0.109823904	0.091837062	16.37789332
34	0.881879127	0.170392442	0.265874664	-56.03665316
35	0.886511581	0.157734073	0.07853505	50.21047213
36	0.894100586	2.032163585	0.072062132	96.45392071
37	0.909989643	0.273472699	0.087993397	67.82369961
38	0.92952342	0.129168257	0.082245985	36.32647331
39	0.933100196	0.184834968	0.104732662	43.33720352
40	0.933204146	0.107399856	0.105381466	1.879322671
41	0.942425394	0.212187311	0.097421455	54.08704925
42	0.964908062	0.230396307	0.088509791	61.58367611
43	0.967711414	0.175614766	0.100170673	42.95999383
44	0.975126961	0.131923888	0.109212637	17.21542014
45	0.980126841	0.154616239	0.102200972	33.9002341
46	0.983925319	0.2178001	0.192140617	11.78120802
47	0.993427783	0.196128032	0.099026417	49.50929973
48	0.993663086	0.202160995	0.110781005	45.2015931
49	0.994500751	0.212591783	0.099426581	53.23122097
50	0.998840156	0.203973147	0.195471202	4.16816865
			Average Cost Difference (%):	**38.09262336**

Table 24. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region V and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.199655741	0.246074451	-23.24937419
52	1.329198658	1.629061571	0.20705453	87.28995062
53	1.652261099	0.560073879	0.27100742	51.61220156
54	1.732050808	0.457472902	0.316354062	30.84747517
55	1.760957841	0.538337897	0.338849036	37.05644022
56	1.820518877	0.587657188	0.425581972	27.57989182
57	1.828533372	0.553294861	0.355852233	35.6848838
58	1.893715218	0.797735588	0.34855268	56.30724202
59	1.900420695	0.758826153	0.373260042	50.81086223
60	1.969644102	0.614058933	0.383024453	37.6241543
61	1.976334082	0.590945736	0.417074613	29.42251928
62	2.002910229	0.55477491	0.4282052	22.814606
63	2.094697427	0.717065336	0.505872474	29.45238764
64	2.112206523	0.700234576	0.580405351	17.11272616
65	2.142249774	1.666815002	0.523829775	68.57301055
66	2.161830466	2.319469947	0.507338008	78.12698504
67	2.209623524	2.902292244	0.420631854	85.50690906
68	2.232543354	1.071572667	0.607187502	43.33678712
69	2.234850662	0.846603261	0.598583155	29.29590722
70	2.315624628	1.227568736	0.585091523	52.33737179
71	2.386793676	1.841384102	0.670151762	63.60608516
72	2.401746454	1.223585099	0.625135675	48.90950575
73	2.445361846	1.830570116	0.696016397	61.97816237
74	2.466706602	0.82745518	0.538976265	34.86338862
75	2.479790488	0.954363069	0.617235241	35.32490303
76	2.498843498	1.194903532	0.674157	43.58063376
77	2.522092914	0.991574364	0.627608887	36.70581753
78	2.551730261	1.652615865	0.71634561	56.65383438
79	2.620165462	12.17892848	0.790493391	93.50933547
80	2.649793028	1.217476411	0.751728818	38.25516356
81	2.674902192	1.30109964	0.871426755	33.02382628
82	2.692003392	1.287323589	0.8295821	35.55760902
83	2.701726462	1.588507231	0.752006334	52.65955867
84	2.714761477	1.594900853	0.840232453	47.31757456
85	2.749451218	1.323251979	0.757936669	42.72166747
86	2.795156227	1.38144323	0.841151363	39.1106819
87	2.808828434	1.289507533	1.002050283	22.29201795

Table 24 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region V and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	1.514179775	0.834135018	44.91175805
89	2.848903585	1.606660278	0.731160186	54.49192366
90	2.851611437	1.21943244	0.911433243	25.25758601
91	2.870713551	1.582247743	0.904632217	42.82613324
92	2.880350027	3.082652688	0.803151131	73.9461038
93	2.896534118	1.271137863	1.25640027	1.159401628
94	2.938218496	1.396067091	0.868626555	37.78045763
95	2.947634879	1.822407132	1.553266664	14.7684051
96	2.95110354	2.400388075	0.964740009	59.80899843
97	2.962064987	1.329361025	1.488073413	-11.93899815
98	2.96452904	2.474384054	0.918501219	62.87960157
99	2.967810632	1.2628239	0.874121328	30.7804257
100	2.985712699	1.290949916	0.968000501	25.01641712
			Average Cost Difference (%):	**41.90601833**

Table 25. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VI and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
1	0.157460926	0.001787112	0.001756035	1.738940351
2	0.293187079	0.005037878	0.004920388	2.33213718
3	0.316221076	0.006311412	0.0060878	3.542977192
4	0.33126114	0.005101999	0.004662632	8.611661092
5	0.365447776	0.007284514	0.00737724	-1.272922766
6	0.388573818	0.008131774	0.00750104	7.756422708
7	0.414682564	0.009815822	0.009198391	6.290166693
8	0.507910302	0.013482154	0.013781099	-2.217341664
9	0.508723927	0.017571308	0.01743703	0.764188787
10	0.564035602	0.020880404	0.020954155	-0.353204254
11	0.568921476	0.020557525	0.019734541	4.003320784
12	0.621993608	0.01865852	0.015146834	18.82081288
13	0.622111707	0.025891535	0.025585309	1.182726688
14	0.652850918	0.023042072	0.023932802	-3.865666935
15	0.67303375	0.021142996	0.018779783	11.17728404

Table 25 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VI and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
16	0.674354319	0.029970584	0.029152984	2.728006034
17	0.683764431	0.021077498	0.020672257	1.922623535
18	0.684739975	0.022408794	0.022692774	-1.267268872
19	0.728289287	0.034512863	0.033811517	2.032128611
20	0.753863006	0.04262121	0.040998754	3.806687817
21	0.753969075	0.035947219	0.034101678	5.134030623
22	0.769293241	0.032891125	0.0299939	8.808531349
23	0.781665858	0.03182285	0.029936885	5.926451621
24	0.805212622	0.046005618	0.045666789	0.736494942
25	0.814479069	0.042395205	0.042805415	-0.967585043
26	0.817586623	0.02630429	0.023865562	9.271215112
27	0.820748706	0.044133785	0.043270799	1.955386346
28	0.833407694	0.042913722	0.043080878	-0.3895156
29	0.834535397	0.051402634	0.051255261	0.286704836
30	0.841453615	0.049812823	0.049420447	0.787700048
31	0.876241566	0.040240983	0.036699145	8.801569929
32	0.87893391	0.058238051	0.05616268	3.563599079
33	0.880603177	0.04989527	0.05018582	-0.582320692
34	0.881879127	0.031929159	0.032982955	-3.300417616
35	0.886511581	0.045672404	0.042879506	6.115067977
36	0.894100586	0.036650603	0.030782172	16.01182656
37	0.909989643	0.046288952	0.044261995	4.378923414
38	0.92952342	0.051760801	0.053652673	-3.655027316
39	0.933100196	0.053189866	0.052229509	1.805525512
40	0.933204146	0.054676087	0.052767209	3.491248693
41	0.942425394	0.053969841	0.051481763	4.610126959
42	0.964908062	0.043131534	0.039325133	8.825099542
43	0.967711414	0.062165861	0.061383382	1.258694961
44	0.975126961	0.070327167	0.072373606	-2.909884475
45	0.980126841	0.056259329	0.052075052	7.437480215
46	0.983925319	0.057841676	0.056535269	2.258591583
47	0.993427783	0.045246154	0.046325056	-2.384517571
48	0.993663086	0.052165783	0.047989905	8.005013569
49	0.994500751	0.063419471	0.062259916	1.828390743
50	0.998840156	0.062108323	0.060126635	3.190696826
			Average Cost Difference (%):	**3.36065564**

Table 26. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VI and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
51	1.168965631	0.070191713	0.068150652	2.9078377
52	1.329198658	0.117361029	0.109460345	6.731948643
53	1.652261099	0.143674911	0.139235204	3.090106102
54	1.732050808	0.179303516	0.173288531	3.354638773
55	1.760957841	0.214096262	0.211875683	1.037187152
56	1.820518877	0.217084647	0.209392151	3.543546722
57	1.828533372	0.229761775	0.223829047	2.582121236
58	1.893715218	0.204129525	0.191760085	6.059603656
59	1.900420695	0.219215456	0.216525443	1.227109244
60	1.969644102	0.20751687	0.187530226	9.631334716
61	1.976334082	0.233608943	0.231516263	0.895804623
62	2.002910229	0.228263976	0.230335222	-0.907390959
63	2.094697427	0.323792476	0.320701762	0.954535557
64	2.112206523	0.233609617	0.217494183	6.898446109
65	2.142249774	0.328475928	0.319032331	2.87497391
66	2.161830466	0.255047446	0.243609678	4.484564972
67	2.209623524	0.23657631	0.216357605	8.546377989
68	2.232543354	0.323913437	0.302221315	6.696888779
69	2.234850662	0.290089866	0.28477596	1.831813683
70	2.315624628	0.333838442	0.334374034	-0.160434653
71	2.386793676	0.407853192	0.413731914	-1.441381919
72	2.401746454	0.370879516	0.35299893	4.821130554
73	2.445361846	0.363535147	0.350316715	3.636080996
74	2.466706602	0.288637436	0.286798041	0.637268558
75	2.479790488	0.303081871	0.285979794	5.642725251
76	2.498843498	0.389289977	0.375051423	3.657570315
77	2.522092914	0.382799393	0.387668618	-1.272004341
78	2.551730261	0.419035057	0.385241011	8.064730043
79	2.620165462	0.342699584	0.281481046	17.86361591
80	2.649793028	0.390258268	0.393872377	-0.926081435
81	2.674902192	0.445186068	0.430648386	3.265529415
82	2.692003392	0.468286689	0.471840375	-0.758869871
83	2.701726462	0.491347485	0.446572784	9.112634549
84	2.714761477	0.501169848	0.497433609	0.745503644
85	2.749451218	0.455737995	0.450722332	1.10055848
86	2.795156227	0.49244978	0.485167592	1.478767706
87	2.808828434	0.468940135	0.467790329	0.245192429

Table 26 (cont.) Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VI and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	0.519508891	0.511291074	1.581843384
89	2.848903585	0.378647326	0.348047163	8.081441586
90	2.851611437	0.507423167	0.489537912	3.524721758
91	2.870713551	0.500022338	0.496462186	0.711998444
92	2.880350027	0.40937772	0.376648086	7.994971868
93	2.896534118	0.589656375	0.58144814	1.392036967
94	2.938218496	0.55303249	0.527127318	4.684204344
95	2.947634879	0.476586988	0.451252678	5.315778801
96	2.95110354	0.456732073	0.439179763	3.843021138
97	2.962064987	0.53606848	0.527688006	1.56332165
98	2.96452904	0.571488323	0.557432346	2.459538829
99	2.967810632	0.499924199	0.466361881	6.713481384
100	2.985712699	0.645541065	0.654191947	-1.340097886
			Average Cost Difference (%):	**3.49360493**

Table 27. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VI and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
1	0.157460926	0.00349229	0.003264837	6.513001516
2	0.293187079	0.015058367	0.008467066	43.77168287
3	0.316221076	0.013889551	0.010672955	23.15838909
4	0.33126114	0.025155954	0.009789764	61.08371046
5	0.365447776	0.018187926	0.012794015	29.65655186
6	0.388573818	0.026858774	0.014510821	45.97362831
7	0.414682564	0.044570523	0.018050496	59.50126989
8	0.507910302	0.191886474	0.032275764	83.17976101
9	0.508723927	0.044473255	0.028064927	36.8948222
10	0.564035602	0.13713821	0.035395664	74.18978703
11	0.568921476	0.08890041	0.037000485	58.37984876
12	0.621993608	0.714279515	0.04448058	93.77266475
13	0.622111707	0.056667494	0.050507944	10.86963571
14	0.652850918	0.07955313	0.041483741	47.85404237
15	0.67303375	0.116051631	0.059718463	48.54146981

Table 27 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VI and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
16	0.674354319	0.089801276	0.046095066	48.66992116
17	0.683764431	0.070920477	0.041553911	41.40773901
18	0.684739975	0.061879376	0.042537097	31.25803803
19	0.728289287	0.137090686	0.059161045	56.84532147
20	0.753863006	0.076017819	0.07290567	4.093973022
21	0.753969075	0.135914651	0.059152943	56.47787627
22	0.769293241	0.091202424	0.059648957	34.5971803
23	0.781665858	0.146675045	0.070940608	51.63416638
24	0.805212622	0.106568052	0.082599594	22.49122271
25	0.814479069	0.099776774	0.081304137	18.51396426
26	0.817586623	0.136041409	0.051932838	61.82571286
27	0.820748706	0.097665928	0.104537157	-7.035441409
28	0.833407694	0.122711451	0.069471373	43.38639741
29	0.834535397	0.096651786	0.081547995	15.62701662
30	0.841453615	0.101273018	0.089838269	11.29101214
31	0.876241566	0.132275	0.090425568	31.63820246
32	0.87893391	0.117986472	0.084291419	28.55840345
33	0.880603177	0.109823904	0.091837062	16.37789332
34	0.881879127	0.170392442	0.265874664	-56.03665316
35	0.886511581	0.157734073	0.07853505	50.21047213
36	0.894100586	2.032163585	0.072062132	96.45392071
37	0.909989643	0.273472699	0.087993397	67.82369961
38	0.92952342	0.129168257	0.082245985	36.32647331
39	0.933100196	0.184834968	0.104732662	43.33720352
40	0.933204146	0.107399856	0.105381466	1.879322671
41	0.942425394	0.212187311	0.097421455	54.08704925
42	0.964908062	0.230396307	0.088509791	61.58367611
43	0.967711414	0.175614766	0.100170673	42.95999383
44	0.975126961	0.131923888	0.109212637	17.21542014
45	0.980126841	0.154616239	0.102200972	33.9002341
46	0.983925319	0.2178001	0.192140617	11.78120802
47	0.993427783	0.196128032	0.099026417	49.50929973
48	0.993663086	0.202160995	0.110781005	45.2015931
49	0.994500751	0.212591783	0.099426581	53.23122097
50	0.998840156	0.203973147	0.195471202	4.16816865
			Average Cost Difference (%):	**38.09262336**

Table 28. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VI and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.199655741	0.246074451	-23.24937419
52	1.329198658	1.629061571	0.20705453	87.28995062
53	1.652261099	0.560073879	0.27100742	51.61220156
54	1.732050808	0.457472902	0.316354062	30.84747517
55	1.760957841	0.538337897	0.338849036	37.05644022
56	1.820518877	0.587657188	0.425581972	27.57989182
57	1.828533372	0.553294861	0.355852233	35.6848838
58	1.893715218	0.797735588	0.34855268	56.30724202
59	1.900420695	0.758826153	0.373260042	50.81086223
60	1.969644102	0.614058933	0.383024453	37.6241543
61	1.976334082	0.590945736	0.417074613	29.42251928
62	2.002910229	0.55477491	0.4282052	22.814606
63	2.094697427	0.717065336	0.505872474	29.45238764
64	2.112206523	0.700234576	0.580405351	17.11272616
65	2.142249774	1.666815002	0.523829775	68.57301055
66	2.161830466	2.319469947	0.507338008	78.12698504
67	2.209623524	2.902292244	0.420631854	85.50690906
68	2.232543354	1.071572667	0.607187502	43.33678712
69	2.234850662	0.846603261	0.598583155	29.29590722
70	2.315624628	1.227568736	0.585091523	52.33737179
71	2.386793676	1.841384102	0.670151762	63.60608516
72	2.401746454	1.223585099	0.625135675	48.90950575
73	2.445361846	1.830570116	0.696016397	61.97816237
74	2.466706602	0.82745518	0.538976265	34.86338862
75	2.479790488	0.954363069	0.617235241	35.32490303
76	2.498843498	1.194903532	0.674157	43.58063376
77	2.522092914	0.991574364	0.627608887	36.70581753
78	2.551730261	1.652615865	0.71634561	56.65383438
79	2.620165462	12.17892848	0.790493391	93.50933547
80	2.649793028	1.217476411	0.751728818	38.25516356
81	2.674902192	1.30109964	0.871426755	33.02382628
82	2.692003392	1.287323589	0.8295821	35.55760902
83	2.701726462	1.588507231	0.752006334	52.65955867
84	2.714761477	1.594900853	0.840232453	47.31757456
85	2.749451218	1.323251979	0.757936669	42.72166747
86	2.795156227	1.38144323	0.841151363	39.1106819
87	2.808828434	1.289507533	1.002050283	22.29201795

Table 28 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VI and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	1.514179775	0.834135018	44.91175805
89	2.848903585	1.606660278	0.731160186	54.49192366
90	2.851611437	1.21943244	0.911433243	25.25758601
91	2.870713551	1.582247743	0.904632217	42.82613324
92	2.880350027	3.082652688	0.803151131	73.9461038
93	2.896534118	1.271137863	1.25640027	1.159401628
94	2.938218496	1.396067091	0.868626555	37.78045763
95	2.947634879	1.822407132	1.553266664	14.7684051
96	2.95110354	2.400388075	0.964740009	59.80899843
97	2.962064987	1.329361025	1.488073413	-11.93899815
98	2.96452904	2.474384054	0.918501219	62.87960157
99	2.967810632	1.2628239	0.874121328	30.7804257
100	2.985712699	1.290949916	0.968000501	25.01641712
			Average Cost Difference (%):	**41.90601833**

Table 29. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VII and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.001787112	0.001756035	1.738940351
2	0.293187079	0.005037878	0.004920388	2.33213718
3	0.316221076	0.006311412	0.0060878	3.542977192
4	0.33126114	0.005101999	0.004662632	8.611661092
5	0.365447776	0.007284514	0.00737724	-1.272922766
6	0.388573818	0.008131774	0.00750104	7.756422708
7	0.414682564	0.009815822	0.009198391	6.290166693
8	0.507910302	0.013482154	0.013781099	-2.217341664
9	0.508723927	0.017571308	0.01743703	0.764188787
10	0.564035602	0.020880404	0.020954155	-0.353204254
11	0.568921476	0.020557525	0.019734541	4.003320784
12	0.621993608	0.01865852	0.015146834	18.82081288
13	0.622111707	0.025891535	0.025585309	1.182726688
14	0.652850918	0.023042072	0.023932802	-3.865666935
15	0.67303375	0.021142996	0.018779783	11.17728404

Table 29 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VII and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
16	0.674354319	0.029970584	0.029152984	2.728006034
17	0.683764431	0.021077498	0.020672257	1.922623535
18	0.684739975	0.022408794	0.022692774	-1.267268872
19	0.728289287	0.034512863	0.033811517	2.032128611
20	0.753863006	0.04262121	0.040998754	3.806687817
21	0.753969075	0.035947219	0.034101678	5.134030623
22	0.769293241	0.032891125	0.0299939	8.808531349
23	0.781665858	0.03182285	0.029936885	5.926451621
24	0.805212622	0.046005618	0.045666789	0.736494942
25	0.814479069	0.042395205	0.042805415	-0.967585043
26	0.817586623	0.02630429	0.023865562	9.271215112
27	0.820748706	0.044133785	0.043270799	1.955386346
28	0.833407694	0.042913722	0.043080878	-0.3895156
29	0.834535397	0.051402634	0.051255261	0.286704836
30	0.841453615	0.049812823	0.049420447	0.787700048
31	0.876241566	0.040240983	0.036699145	8.801569929
32	0.87893391	0.058238051	0.05616268	3.563599079
33	0.880603177	0.04989527	0.05018582	-0.582320692
34	0.881879127	0.031929159	0.032982955	-3.300417616
35	0.886511581	0.045672404	0.042879506	6.115067977
36	0.894100586	0.036650603	0.030782172	16.01182656
37	0.909989643	0.046288952	0.044261995	4.378923414
38	0.92952342	0.051760801	0.053652673	-3.655027316
39	0.933100196	0.053189866	0.052229509	1.805525512
40	0.933204146	0.054676087	0.052767209	3.491248693
41	0.942425394	0.053969841	0.051481763	4.610126959
42	0.964908062	0.043131534	0.039325133	8.825099542
43	0.967711414	0.062165861	0.061383382	1.258694961
44	0.975126961	0.070327167	0.072373606	-2.909884475
45	0.980126841	0.056259329	0.052075052	7.437480215
46	0.983925319	0.057841676	0.056535269	2.258591583
47	0.993427783	0.045246154	0.046325056	-2.384517571
48	0.993663086	0.052165783	0.047989905	8.005013569
49	0.994500751	0.063419471	0.062259916	1.828390743
50	0.998840156	0.062108323	0.060126635	3.190696826
			Average Cost Difference (%):	**3.36065564**

Table 30. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.072919536	0.069778784	4.307147217
52	1.329198658	0.11362718	0.108816334	4.233886525
53	1.652261099	0.159482321	0.137926034	13.51641168
54	1.732050808	0.181999412	0.178882594	1.712542834
55	1.760957841	0.212423498	0.210689533	0.81627715
56	1.820518877	0.21546605	0.207444698	3.722791425
57	1.828533372	0.238020329	0.225666074	5.190420403
58	1.893715218	0.192082563	0.191161808	0.479354045
59	1.900420695	0.222392404	0.213270569	4.101684854
60	1.969644102	0.194592541	0.185107048	4.87454053
61	1.976334082	0.238466274	0.226449741	5.039091209
62	2.002910229	0.237799276	0.227285708	4.421194114
63	2.094697427	0.313407755	0.317698041	-1.36891525
64	2.112206523	0.23064719	0.207205359	10.16350162
65	2.142249774	0.327231214	0.320043158	2.196628845
66	2.161830466	0.249882207	0.253855732	-1.590159171
67	2.209623524	0.232959658	0.204951073	12.02293393
68	2.232543354	0.304115102	0.309437035	-1.749973326
69	2.234850662	0.295887363	0.278467041	5.887484514
70	2.315624628	0.342873677	0.332544591	3.012504882
71	2.386793676	0.408063911	0.40235781	1.398335235
72	2.401746454	0.360819165	0.361294624	-0.131771955
73	2.445361846	0.354098169	0.348719635	1.51893867
74	2.466706602	0.322296533	0.279987417	13.12738781
75	2.479790488	0.33032071	0.297281537	10.00215002
76	2.498843498	0.390811901	0.366240044	6.287387052
77	2.522092914	0.402586537	0.377726641	6.175043904
78	2.551730261	0.394234241	0.383648837	2.68505451
79	2.620165462	0.292008017	0.283587524	2.883651289
80	2.649793028	0.415305178	0.386707215	6.886011706
81	2.674902192	0.439758875	0.446670939	-1.571785007
82	2.692003392	0.472968023	0.46077734	2.577485616
83	2.701726462	0.46026769	0.459636071	0.13722847
84	2.714761477	0.503729031	0.491721916	2.383645569
85	2.749451218	0.465110721	0.453033806	2.596567582
86	2.795156227	0.524017238	0.488921663	6.697408575
87	2.808828434	0.48910839	0.4929749	-0.790522228

Table 30 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	0.519540378	0.49987758	3.784652475
89	2.848903585	0.414397872	0.406496304	1.906758765
90	2.851611437	0.528775079	0.487798297	7.74937842
91	2.870713551	0.480934024	0.488490598	-1.571228639
92	2.880350027	0.405485879	0.373471784	7.895242726
93	2.896534118	0.582987434	0.580419748	0.440435911
94	2.938218496	0.515189384	0.528103986	-2.506767818
95	2.947634879	0.483742987	0.447703671	7.450095866
96	2.95110354	0.440004398	0.441918159	-0.434941361
97	2.962064987	0.524340021	0.519175852	0.984889305
98	2.96452904	0.563295323	0.547561207	2.793226855
99	2.967810632	0.548037076	0.483594644	11.75877218
100	2.985712699	0.651990494	0.63486768	2.626236813
			Average Cost Difference (%):	**3.734566327**

Table 31. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VII and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
1	0.157460926	0.00349229	0.003264837	6.513001516
2	0.293187079	0.015058367	0.008467066	43.77168287
3	0.316221076	0.013889551	0.010672955	23.15838909
4	0.33126114	0.025155954	0.009789764	61.08371046
5	0.365447776	0.018187926	0.012794015	29.65655186
6	0.388573818	0.026858774	0.014510821	45.97362831
7	0.414682564	0.044570523	0.018050496	59.50126989
8	0.507910302	0.191886474	0.032275764	83.17976101
9	0.508723927	0.044473255	0.028064927	36.8948222
10	0.564035602	0.13713821	0.035395664	74.18978703
11	0.568921476	0.08890041	0.037000485	58.37984876
12	0.621993608	0.714279515	0.04448058	93.77266475
13	0.622111707	0.056667494	0.050507944	10.86963571
14	0.652850918	0.07955313	0.041483741	47.85404237
15	0.67303375	0.116051631	0.059718463	48.54146981

Table 31 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VII and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
16	0.674354319	0.089801276	0.046095066	48.66992116
17	0.683764431	0.070920477	0.041553911	41.40773901
18	0.684739975	0.061879376	0.042537097	31.25803803
19	0.728289287	0.137090686	0.059161045	56.84532147
20	0.753863006	0.076017819	0.07290567	4.093973022
21	0.753969075	0.135914651	0.059152943	56.47787627
22	0.769293241	0.091202424	0.059648957	34.5971803
23	0.781665858	0.146675045	0.070940608	51.63416638
24	0.805212622	0.106568052	0.082599594	22.49122271
25	0.814479069	0.099776774	0.081304137	18.51396426
26	0.817586623	0.136041409	0.051932838	61.82571286
27	0.820748706	0.097665928	0.104537157	-7.035441409
28	0.833407694	0.122711451	0.069471373	43.38639741
29	0.834535397	0.096651786	0.081547995	15.62701662
30	0.841453615	0.101273018	0.089838269	11.29101214
31	0.876241566	0.132275	0.090425568	31.63820246
32	0.87893391	0.117986472	0.084291419	28.55840345
33	0.880603177	0.109823904	0.091837062	16.37789332
34	0.881879127	0.170392442	0.265874664	-56.03665316
35	0.886511581	0.157734073	0.07853505	50.21047213
36	0.894100586	2.032163585	0.072062132	96.45392071
37	0.909989643	0.273472699	0.087993397	67.82369961
38	0.92952342	0.129168257	0.082245985	36.32647331
39	0.933100196	0.184834968	0.104732662	43.33720352
40	0.933204146	0.107399856	0.105381466	1.879322671
41	0.942425394	0.212187311	0.097421455	54.08704925
42	0.964908062	0.230396307	0.088509791	61.58367611
43	0.967711414	0.175614766	0.100170673	42.95999383
44	0.975126961	0.131923888	0.109212637	17.21542014
45	0.980126841	0.154616239	0.102200972	33.9002341
46	0.983925319	0.2178001	0.192140617	11.78120802
47	0.993427783	0.196128032	0.099026417	49.50929973
48	0.993663086	0.202160995	0.110781005	45.2015931
49	0.994500751	0.212591783	0.099426581	53.23122097
50	0.998840156	0.203973147	0.195471202	4.16816865
			Average Cost Difference (%):	**38.09262336**

Table 32. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
51	1.168965631	0.285582285	0.144519534	49.39478338
52	1.329198658	0.269827876	0.201766684	25.22392894
53	1.652261099	6.3961436	0.30205764	95.27750378
54	1.732050808	0.971567466	0.379595395	60.92958975
55	1.760957841	0.487998511	0.356113483	27.02570297
56	1.820518877	0.835525381	0.356483017	57.33426832
57	1.828533372	0.756925993	0.363328782	51.99943125
58	1.893715218	0.637495568	0.378880163	40.56740441
59	1.900420695	1.387698276	0.404050304	70.88341817
60	1.969644102	0.792771058	0.403733072	49.07318227
61	1.976334082	0.738670275	0.395742341	46.42503503
62	2.002910229	0.721830057	0.411381322	43.00856295
63	2.094697427	0.865749627	0.552531228	36.17886622
64	2.112206523	5.26063037	0.47079377	91.05062062
65	2.142249774	0.825728315	0.551985841	33.15163945
66	2.161830466	0.737989617	0.447537233	39.35724529
67	2.209623524	0.993018734	0.573809098	42.21568253
68	2.232543354	0.635390196	0.508653755	19.94623796
69	2.234850662	1.501862368	0.54415052	63.76828315
70	2.315624628	0.894141083	0.571382485	36.0970549
71	2.386793676	1.162220203	0.596587881	48.6682576
72	2.401746454	3.576443304	0.673407199	81.17103664
73	2.445361846	0.892904387	0.679292194	23.92329976
74	2.466706602	0.79146168	0.669743425	15.37891951
75	2.479790488	1.801543661	0.557929502	69.03047567
76	2.498843498	1.412336426	0.691577578	51.03308495
77	2.522092914	0.881344308	0.709236934	19.52782496
78	2.551730261	40.33179831	0.695687471	98.27508938
79	2.620165462	2.002149473	0.970655543	51.51932677
80	2.649793028	43.83578307	0.742442739	98.30630894
81	2.674902192	0.983466935	0.758767019	22.84773472
82	2.692003392	1.597268619	0.79637272	50.14159104
83	2.701726462	23.5962763	0.872479102	96.30247124
84	2.714761477	1.205836026	0.891077268	26.10294859
85	2.749451218	1.537852656	0.940839667	38.82120873
86	2.795156227	5.119470286	0.897539983	82.46810836
87	2.808828434	1.540931051	0.917962594	40.4280553

Table 32 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
88	2.813805571	1.555741064	0.896475474	42.37630572
89	2.848903585	3.093538602	0.793976151	74.33437066
90	2.851611437	5.770058687	0.914681711	84.14779188
91	2.870713551	1.597128785	0.876549096	45.1171938
92	2.880350027	1.214974365	0.759084095	37.52262456
93	2.896534118	1.287866823	0.955770589	25.7865354
94	2.938218496	1.599258348	0.950804332	40.54717089
95	2.947634879	2.792630089	0.876430113	68.61631921
96	2.95110354	8.102297387	0.809516388	90.00880431
97	2.962064987	1.660226278	0.958689355	42.25550046
98	2.96452904	1.259679949	0.938393954	25.50536709
99	2.967810632	2.157063088	0.980291203	54.5543564
100	2.985712699	1.280137055	0.999812228	21.89803243
			Average Cost Difference (%):	**50.91049113**

Table 33. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VIII and Initial Conditions within Norm 1

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
1	0.157460926	0.001792059	0.001759754	1.802667094
2	0.293187079	0.005057754	0.00491277	2.866571796
3	0.316221076	0.00639192	0.006257919	2.096405729
4	0.33126114	0.005392618	0.004573353	15.19234241
5	0.365447776	0.007652993	0.007106402	7.142181171
6	0.388573818	0.008051113	0.007547189	6.259064808
7	0.414682564	0.009485212	0.009692929	-2.189901982
8	0.507910302	0.014324566	0.013050842	8.891889061
9	0.508723927	0.017595158	0.017155858	2.49670737
10	0.564035602	0.021031315	0.020785368	1.169432105
11	0.568921476	0.020253885	0.019958606	1.457888891
12	0.621993608	0.018560131	0.0151625	18.30607105
13	0.622111707	0.026346711	0.025567119	2.958974181
14	0.652850918	0.025241288	0.024562962	2.68736816
15	0.67303375	0.021443811	0.019679741	8.226475496

Table 33 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VIII and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
16	0.674354319	0.028839268	0.029285314	-1.546661785
17	0.683764431	0.021708392	0.020410631	5.978154918
18	0.684739975	0.023057336	0.023026123	0.135372092
19	0.728289287	0.03404093	0.033115056	2.719886379
20	0.753863006	0.040499612	0.041570551	-2.644319019
21	0.753969075	0.0360001	0.034458823	4.281314222
22	0.769293241	0.031734128	0.030435905	4.090936463
23	0.781665858	0.033025176	0.030670347	7.130405751
24	0.805212622	0.046087491	0.045403745	1.48358321
25	0.814479069	0.042750628	0.041298158	3.397540882
26	0.817586623	0.028017149	0.026977594	3.710423737
27	0.820748706	0.044573634	0.04271126	4.17819482
28	0.833407694	0.043284073	0.043095174	0.43641797
29	0.834535397	0.052508495	0.051783953	1.37985751
30	0.841453615	0.049917355	0.048807585	2.223214555
31	0.876241566	0.038352299	0.036474553	4.896046264
32	0.87893391	0.058460693	0.057566862	1.528944012
33	0.880603177	0.052737688	0.050390569	4.45055504
34	0.881879127	0.032363909	0.029602387	8.532722638
35	0.886511581	0.042967073	0.041978322	2.301183288
36	0.894100586	0.032492583	0.033170626	-2.086762546
37	0.909989643	0.04511822	0.042867434	4.988639796
38	0.92952342	0.052887669	0.051815727	2.026827304
39	0.933100196	0.053438195	0.055392766	-3.657629322
40	0.933204146	0.05309186	0.05393725	-1.592314616
41	0.942425394	0.054498071	0.052538842	3.595043343
42	0.964908062	0.042093549	0.040505133	3.773536198
43	0.967711414	0.062424927	0.06049442	3.092525483
44	0.975126961	0.072247687	0.070069242	3.015245883
45	0.980126841	0.052972622	0.051496408	2.786749117
46	0.983925319	0.057631003	0.054082397	6.157459276
47	0.993427783	0.047723815	0.044037393	7.724492029
48	0.993663086	0.05326616	0.05089677	4.44820913
49	0.994500751	0.062572046	0.059360566	5.132451541
50	0.998840156	0.064179525	0.059896043	6.674218901
			Average Cost Difference (%):	**3.682132036**

Table 34. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VIII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
51	1.168965631	0.070245779	0.068230392	2.869050437
52	1.329198658	0.117463472	0.109469492	6.805502619
53	1.652261099	0.143895357	0.139387729	3.132573433
54	1.732050808	0.179550232	0.173456861	3.393686053
55	1.760957841	0.214270745	0.211979152	1.069485073
56	1.820518877	0.217240747	0.209549624	3.540368458
57	1.828533372	0.229874129	0.223959981	2.572776917
58	1.893715218	0.204247576	0.191921918	6.034665388
59	1.900420695	0.219421845	0.216777136	1.205308088
60	1.969644102	0.207810879	0.187859942	9.600525839
61	1.976334082	0.233784174	0.2316903	0.89564389
62	2.002910229	0.228606799	0.230551107	-0.850503325
63	2.094697427	0.323991961	0.32081492	0.980592548
64	2.112206523	0.233710265	0.217858885	6.782492004
65	2.142249774	0.328948678	0.319311361	2.929732876
66	2.161830466	0.255456614	0.244001061	4.484344016
67	2.209623524	0.236789592	0.216612669	8.521034642
68	2.232543354	0.324032225	0.302356778	6.689287309
69	2.234850662	0.29042297	0.28514921	1.815889347
70	2.315624628	0.334264138	0.334716973	-0.135472379
71	2.386793676	0.407934943	0.414111166	-1.514021564
72	2.401746454	0.371028588	0.353334556	4.76891342
73	2.445361846	0.363890478	0.351063752	3.524886403
74	2.466706602	0.288768335	0.287397648	0.474666648
75	2.479790488	0.303431366	0.286379485	5.619683026
76	2.498843498	0.389911084	0.375446457	3.709724499
77	2.522092914	0.383458185	0.3879817	-1.179663181
78	2.551730261	0.41964923	0.385124659	8.227006941
79	2.620165462	0.343109175	0.281709368	17.89512256
80	2.649793028	0.390550632	0.394715022	-1.066286887
81	2.674902192	0.445465459	0.431014817	3.243942234
82	2.692003392	0.468729162	0.472361333	-0.774897567
83	2.701726462	0.49334873	0.447548344	9.283572398
84	2.714761477	0.501847977	0.497888644	0.788950623
85	2.749451218	0.456739879	0.451158763	1.221946398
86	2.795156227	0.492913987	0.485767194	1.449906688
87	2.808828434	0.469482248	0.468130921	0.28783342

Table 34 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.14$ in Region VIII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	0.520561947	0.51202472	1.64000222
89	2.848903585	0.378694632	0.34827571	8.032572841
90	2.851611437	0.508560987	0.490061346	3.637644536
91	2.870713551	0.500184949	0.496479765	0.740762916
92	2.880350027	0.410602429	0.377497013	8.062645027
93	2.896534118	0.590504234	0.582117233	1.420311651
94	2.938218496	0.553349084	0.527290152	4.709311392
95	2.947634879	0.476814481	0.451724599	5.261979876
96	2.95110354	0.457190313	0.439490715	3.871385165
97	2.962064987	0.536774667	0.528172765	1.602516378
98	2.96452904	0.572060067	0.558076997	2.444335972
99	2.967810632	0.501318354	0.466710084	6.903451591
100	2.985712699	0.645811062	0.654568747	-1.356075395
			Average Cost Difference (%):	**3.505382269**

Table 35. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VIII and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
1	0.157460926	0.004535157	0.002775515	38.80002783
2	0.293187079	0.021782673	0.012298881	43.53823724
3	0.316221076	0.021171265	0.010247173	51.59867302
4	0.33126114	0.018644523	0.009913928	46.82659518
5	0.365447776	0.037648693	0.0130614	65.30716255
6	0.388573818	0.060974891	0.014280811	76.57919373
7	0.414682564	0.086295561	0.063836421	26.0258342
8	0.507910302	0.047218653	0.035762877	24.26112444
9	0.508723927	0.046216553	0.029565503	36.02832536
10	0.564035602	0.049114322	0.035170456	28.39063199
11	0.568921476	0.096893833	0.038697096	60.06237447
12	0.621993608	0.066911313	0.032344855	51.66010943
13	0.622111707	0.06335594	0.045047073	28.89842243
14	0.652850918	0.100312001	0.067214707	32.99435057
15	0.67303375	0.060459975	0.041662937	31.09005312

Table 35 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VIII and Initial Conditions within Norm 1

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
16	0.674354319	0.148568656	0.05115695	65.56679475
17	0.683764431	0.405859829	0.056358457	86.11381243
18	0.684739975	0.140313976	0.058758306	58.12369695
19	0.728289287	0.09501592	0.06316325	33.52350878
20	0.753863006	0.079549933	0.065099894	18.1647406
21	0.753969075	0.092280044	0.0623443	32.4401054
22	0.769293241	0.248992727	0.06714767	73.03227671
23	0.781665858	0.110461212	0.053588294	51.48677709
24	0.805212622	0.102104076	0.085311683	16.44634931
25	0.814479069	0.089898883	0.304521187	-238.7374544
26	0.817586623	0.402745525	0.087429983	78.29150731
27	0.820748706	0.15431882	0.08037901	47.91367021
28	0.833407694	0.109105783	0.077525204	28.94491802
29	0.834535397	0.120426456	0.082387527	31.58685379
30	0.841453615	0.134263746	0.079972883	40.4359813
31	0.876241566	0.277277858	0.08225977	70.33309091
32	0.87893391	0.1450879	0.093537008	35.53080022
33	0.880603177	0.172683348	0.090259041	47.73147346
34	0.881879127	0.674305647	0.182434881	72.94477934
35	0.886511581	0.363956253	0.081616075	77.57530639
36	0.894100586	0.148024893	0.072536534	50.99707058
37	0.909989643	0.360982273	0.097809395	72.90465416
38	0.92952342	0.126124495	0.100391057	20.40320397
39	0.933100196	0.138127527	0.115203111	16.59655858
40	0.933204146	0.129727611	0.147939099	-14.03825117
41	0.942425394	0.138735926	0.150439717	-8.436019877
42	0.964908062	0.298877097	0.080800388	72.96534642
43	0.967711414	0.196230157	0.101718692	48.16357796
44	0.975126961	0.15381182	0.126168375	17.97224979
45	0.980126841	0.117646671	0.099730071	15.22915978
46	0.983925319	0.156219116	0.136568171	12.57909112
47	0.993427783	0.274679565	0.100453065	63.42899965
48	0.993663086	0.220484906	0.10733801	51.31729763
49	0.994500751	0.415781822	0.124283905	70.10838412
50	0.998840156	0.16546028	0.112362944	32.09068426
			Average Cost Difference (%):	37.83584222

Table 36. Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	Deterministic Linear	Stochastic Nonlinear	Cost Error (%)
51	1.168965631	0.199268285	0.247438501	-24.17354875
52	1.329198658	1.642395547	0.207551646	87.36287089
53	1.652261099	0.563090231	0.271914883	51.71024674
54	1.732050808	0.457029415	0.318240024	30.36771516
55	1.760957841	0.53752857	0.339747365	36.79454751
56	1.820518877	0.589690053	0.434604814	26.29944983
57	1.828533372	0.549328134	0.355961554	35.20056017
58	1.893715218	0.801629778	0.349667461	56.38043011
59	1.900420695	0.760962932	0.372901251	50.99613453
60	1.969644102	0.619672077	0.383674592	38.08425358
61	1.976334082	0.610976003	0.417999149	31.58501357
62	2.002910229	0.558335626	0.428867605	23.18820705
63	2.094697427	0.715095973	0.508476819	28.8939054
64	2.112206523	0.697004219	0.585418469	16.00933636
65	2.142249774	1.671752979	0.52307355	68.71107412
66	2.161830466	2.339005584	0.508634741	78.25423143
67	2.209623524	2.913149444	0.422056995	85.5120033
68	2.232543354	1.073729951	0.608070775	43.36836988
69	2.234850662	0.848757895	0.600479439	29.25197603
70	2.315624628	1.239570132	0.586496239	52.68551381
71	2.386793676	1.879775141	0.675750158	64.05154302
72	2.401746454	1.215872837	0.628484547	48.31001006
73	2.445361846	1.962943323	0.697650601	64.4589534
74	2.466706602	0.83300508	0.540892673	35.06730199
75	2.479790488	0.930137753	0.62152214	33.17955985
76	2.498843498	1.199036	0.674847593	43.71748698
77	2.522092914	0.944717216	0.631720113	33.1313008
78	2.551730261	1.666905421	0.71847658	56.89757972
79	2.620165462	12.21159167	0.791960209	93.51468482
80	2.649793028	1.239420256	0.750476421	39.4493984
81	2.674902192	1.30986219	0.870943803	33.50874543
82	2.692003392	1.284168099	0.791930689	38.33122862
83	2.701726462	1.605677835	0.755005209	52.97903525
84	2.714761477	1.616452949	0.846707519	47.61941447
85	2.749451218	1.351313993	0.761567924	43.64241563
86	2.795156227	1.39371587	0.846905944	39.23395992
87	2.808828434	1.297656749	1.010098132	22.15983677

Table 36 (cont.). Mean Cost Comparison of the Linear Deterministic and the Stochastic Nonlinear Control with $\varepsilon = 0.28$ in Region VII and Initial Conditions Between Norm 1 and 3

	Initial Condition Norm	**Deterministic Linear**	**Stochastic Nonlinear**	**Cost Error (%)**
88	2.813805571	1.525303493	0.835964752	45.19354637
89	2.848903585	1.610770268	0.727689055	54.82353567
90	2.851611437	1.22319458	0.916860317	25.04378844
91	2.870713551	1.588145517	0.90619685	42.93993589
92	2.880350027	3.114680512	0.810668377	73.97266352
93	2.896534118	1.288463281	1.300779489	-0.955883504
94	2.938218496	1.400484588	0.871475599	37.77328174
95	2.947634879	1.83657069	1.600178365	12.87139815
96	2.95110354	2.428432578	0.961442048	60.40894624
97	2.962064987	1.31687053	1.508836773	-14.57745758
98	2.96452904	2.622738637	0.920474722	64.90406217
99	2.967810632	1.274000862	0.876523724	31.19912634
100	2.985712699	1.340235132	0.96812078	27.76485586
			Average Cost Difference (%):	**41.9419309**