Forward-backward Stochastic Differential Equations in stochastic optimal control, Backward Doubly Stochastic Differential Equations in filtering

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Itô and Stratonovich integrals

Partition $[0, t]$ into N intervals $[t_0, t_1), [t_1, t_2), \ldots, [t_{N-1}, t_N)$, where $t_k = k \left\lfloor \frac{t}{N} \right\rfloor$.

For $s \in [0, t]$, let $H_s(\omega) = \sum_k H_k(\omega)\chi_{[s_k, s_{k+1})}(s)$ be a $\mathcal{F}_s$-measurable simple function.

- Itô integral:

$$\int_0^t H_s dB_s = \lim_{N \searrow \infty} \sum_{k=1}^{N} H_k \left(B_{t_{k+1}} - B_{t_k}\right)$$

- Stratonovich integral:

$$\int_0^t H_s \circ dB_s = \lim_{N \searrow \infty} \sum_{k=1}^{N} \frac{1}{2} \left(H_{k+1} + H_k\right) \left(B_{t_{k+1}} - B_{t_k}\right)$$

The material presented in these notes are for heuristical purposes; they are nowhere close to proofs.

For rigorous treatment, please refer to the references cited here and references therein.

- Itô stochastic differential equation:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s$$
$$\rightarrow \quad dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

- Stratonovich stochastic differential equation:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s) \circ dB_s$$
$$\rightarrow \quad dX_t = b(X_t)dt + \sigma(X_t) \circ dB_t$$

Itô-Taylor expansion

$$
\begin{aligned}
&\varphi(X_{t+\Delta t})\\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(X_{t+\Delta t} - X_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)(X_{t+\Delta t} - X_t)^2 + \mathcal{O}((X_{t+\Delta t} - X_t)^3)\\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(b(X_t)\Delta t + \sigma(X_t)\Delta W_t)\\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)(b(X_t)^2(\Delta t)^2 + 2b(X_t)\sigma(X_t)\underbrace{\Delta t \Delta W_t}_{\sim \Delta t^{3/2}} + \sigma(X_t)^2\underbrace{(\Delta W_t)^2}_{\sim \Delta t})\\
&\quad + \mathcal{O}((X_{t+\Delta t} - X_t)^3)\\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(b(X_t)\Delta t + \sigma(X_t)\Delta W_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)\sigma(X_t)^2\Delta t + \mathcal{O}(\Delta t^{3/2})
\end{aligned}
$$

Itô-Taylor expansion

$$
\begin{aligned}
&\varphi(X_{t+\Delta t}) \\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(X_{t+\Delta t} - X_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)(X_{t+\Delta t} - X_t)^2 + \mathcal{O}((X_{t+\Delta t} - X_t)^3) \\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(b(X_t)\Delta t + \sigma(X_t)\Delta W_t) \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)(b(X_t)^2(\Delta t)^2 + 2b(X_t)\sigma(X_t)\underbrace{\Delta t \Delta W_t}_{\sim \Delta t^{3/2}} + \sigma(X_t)^2 \underbrace{(\Delta W_t)^2}_{\sim \Delta t}) \\
&\quad + \mathcal{O}((X_{t+\Delta t} - X_t)^3) \\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(b(X_t)\Delta t + \sigma(X_t)\Delta W_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)\sigma(X_t)^2\Delta t + \mathcal{O}(\Delta t^{3/2})
\end{aligned}
$$

Itô's lemma: $d\varphi(X_t) = \frac{\partial}{\partial x}\varphi(X_t)dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)\sigma(X_t)^2 dt$

Itô-Taylor expansion

$$
\begin{aligned}
&\varphi(X_{t+\Delta t}) \\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(X_{t+\Delta t} - X_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)(X_{t+\Delta t} - X_t)^2 + \mathcal{O}((X_{t+\Delta t} - X_t)^3) \\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(b(X_t)\Delta t + \sigma(X_t)\Delta W_t) \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)(b(X_t)^2(\Delta t)^2 + 2b(X_t)\sigma(X_t)\underbrace{\Delta t \Delta W_t}_{\sim \Delta t^{3/2}} + \sigma(X_t)^2\underbrace{(\Delta W_t)^2}_{\sim \Delta t}) \\
&\quad + \mathcal{O}((X_{t+\Delta t} - X_t)^3) \\
&= \varphi(X_t) + \frac{\partial}{\partial x}\varphi(X_t)(b(X_t)\Delta t + \sigma(X_t)\Delta W_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)\sigma(X_t)^2\Delta t + \mathcal{O}(\Delta t^{3/2})
\end{aligned}
$$

Generator of Itô diffusion:

$$
\begin{aligned}
\mathcal{L}\varphi(x) &= \lim_{\Delta t \searrow 0}\frac{\mathbb{E}[\varphi(X_{t+\Delta t})|X_t = x] - \varphi(x)}{\Delta t} = \lim_{\Delta t \searrow 0}\frac{\mathbb{E}[\varphi(X_{t+\Delta t}) - \varphi(x)X_t = x]}{\Delta t} \\
&= \frac{\partial}{\partial x}\varphi(X_t)(b(X_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(X_t)\sigma(X_t)^2
\end{aligned}
$$

Relation between Itô and Stratonovich integrals

For $\mathcal{F}_t^B$-measurable function $\varphi : \Omega \to \mathbb{R}$,

$$\begin{aligned}
\varphi_t(\omega) \circ dB_t &\approx \frac{1}{2}\left(\varphi_{t+h}(\omega) + \varphi_t(\omega)\right)\left(B_{t+h} - B_t\right) \\
&= \frac{1}{2}\left(\varphi_{t+h}(\omega) + \varphi_t(\omega) + \varphi_t(\omega) - \varphi_t(\omega)\right)\left(B_{t+h} - B_t\right) \\
&= \left[\frac{1}{2}\left(\varphi_{t+h}(\omega) - \varphi_t(\omega)\right)\left(B_{t+h} - B_t\right) + \underbrace{\varphi_t(\omega)\left(B_{t+h} - B_t\right)}_{\text{Itô}}\right] \\
&= \frac{1}{2}\left\langle d\varphi(\omega), dB\right\rangle_t + \varphi_t(\omega) dB_t
\end{aligned}$$

Relation between Itô and Stratonovich integrals

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t) \circ dB_t, \\ &= b(X_t)dt + \frac{1}{2}\langle d\sigma(X), dB\rangle_t + \sigma(X_t)dB_t \end{aligned}$$

By Itô's lemma:

$$\begin{aligned} d\sigma(X_t) &= \frac{\partial\sigma(X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2\sigma(X_t)}{\partial x^2}\sigma(X_t)^2 dt \\ &= \frac{\partial\sigma(X_t)}{\partial x}b(X_t)dt + \frac{\partial\sigma(X_t)}{\partial x}\sigma(X_t)dB_t + \frac{1}{2}\frac{\partial^2\sigma(X_t)}{\partial x^2}\sigma(X_t)^2 dt \end{aligned}$$

Then,

$$\langle d\sigma(X), dB\rangle_t = \frac{\partial\sigma(X_t)}{\partial x}\sigma(X_t)dt,$$

so

$$\sigma(X_t) \circ dB_t = \frac{1}{2}\frac{\partial\sigma(X_t)}{\partial x}\sigma(X_t)dt + \sigma(X_t)dB_t$$

Itô's lemma

$$\begin{aligned}\text{Itô}: \quad & dX_t = \left(b(X_t) + \frac{1}{2}\sigma_x(X_t)\sigma(X_t)\right)dt + \sigma(X_t)dB_t,\\ \text{Stratonovich}: \quad & dX_t = b(X_t)dt + \sigma(X_t)\circ dB_t,\end{aligned}$$

Applying Itô's lemma:

$$\begin{aligned}& d\varphi(X_t)\\ & = \varphi_x(X_t)dX_t + \frac{1}{2}\varphi_{xx}(X_t)\sigma(X_t)^2 dt\\ & = \varphi_x(X_t)\left(b(X_t) + \frac{1}{2}\sigma_x(X_t)\sigma(X_t)\right)dt + \varphi_x(X_t)\sigma(X_t)dB_t + \frac{1}{2}\varphi_{xx}(X_t)\sigma(X_t)^2 dt\\ & = \left(\varphi_x(X_t)b(X_t) + \frac{1}{2}\varphi_x(X_t)\sigma_x(X_t)\sigma(X_t)\right)dt - \frac{1}{2}\left\langle d\left(\varphi_x(X)\sigma(X)\right), dB\right\rangle_t\\ & \quad + \varphi_x(X_t)\sigma(X_t)\circ dB_t + \frac{1}{2}\varphi_{xx}(X_t)\sigma(X_t)^2 dt\end{aligned}$$

and

$$d\left(\varphi_x(X_t)\sigma(X_t)\right) = \left[\varphi_{xx}(X)\sigma(X_t) + \varphi_x(X_t)\sigma_x(X_t)\right]\sigma(X_t)dB_t + (\ldots)\,dt.$$

Itô's lemma

$$\text{Itô}: \quad dX_t = \left(b(X_t) + \frac{1}{2}\sigma_x(X_t)\sigma(X_t) \right) dt + \sigma(X_t) dB_t,$$
$$\text{Stratonovich}: \quad dX_t = b(X_t)dt + \sigma(X_t) \circ dB_t,$$

Applying Itô's lemma:

$$d\varphi(X_t)$$
$$= \left(\varphi_x(X_t)b(X_t) + \frac{1}{2}\varphi_x(X_t)\sigma_x(X_t)\sigma(X_t) \right) dt - \frac{1}{2}\left\langle d\left(\varphi_x(X)\sigma(X)\right), dB\right\rangle_t$$
$$+ \varphi_x(X_t)\sigma(X_t) \circ dB_t + \frac{1}{2}\varphi_{xx}(X_t)\sigma(X_t)^2 dt$$

and

$$d\left(\varphi_x(X_t)\sigma(X_t)\right) = \left[\varphi_{xx}(X)\sigma(X_t) + \varphi_x(X_t)\sigma_x(X_t)\right]\sigma(X_t)dB_t + (\ldots)\,dt.$$

Then,

$$d\varphi(X_t) = \varphi_x(X_t)b(X_t)dt + \varphi_x(X_t)\sigma(X_t) \circ dB_t$$

Backward Itô integral [1]

Define

$$\mathcal{F}_{t,s}^{0,B} \overset{\text{def}}{=} \bigcap_{r<t} \sigma\left(B_u - B_r : r \leq u \leq s\right)$$

and $\mathcal{F}_{t,s}^{B}$ is the completion of $\mathcal{F}_{t,s}^{0,B}$.

Partition $[t, T]$ into N subintervals.
For $s \in [t, T]$, let $H_s = \sum_{k=1}^{N} H_k \chi_{[t_{k-1}, t_k)}(s)$ where $H_k \in \mathcal{F}_{t_{k-1}, T}$.

- Backward Itô integral:

$$\int_t^T H_s d\overleftarrow{B}_s = \lim_{N\to\infty} \sum_{k=1}^{N} H_k (B_{t_k} - B_{t_{k-1}}).$$

[1] Perkowski, Backward Stochastic Differential Equations: An Introduction

Relation between forward and backward Itô integrals:

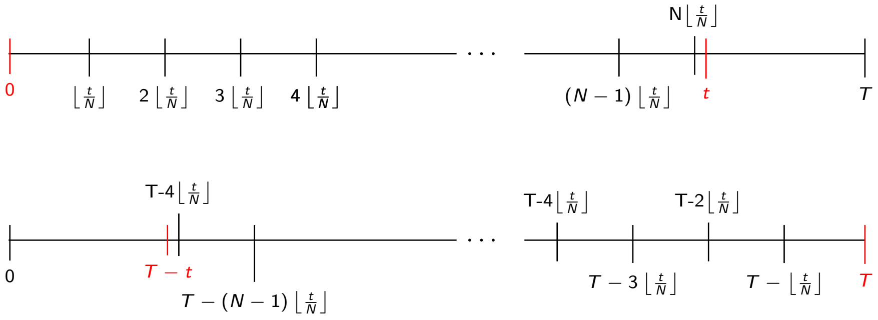
Consider interval $(T-t, T]$.
For $s \in (T-t, T]$, $H'_s := H_{T-s}$ and $B'_s := B_T - B_{T-s}$, we have that

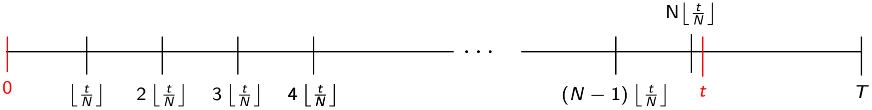
$$\int_{T-t}^{T} H_s d\overleftarrow{B}_s = \int_0^t H'_s dB'_s.$$

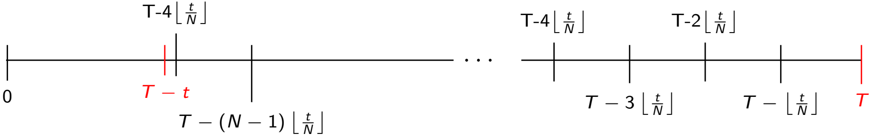
For $s \in (T-t, T]$, $H'_s := H_{T-s}$ and $B'_s := B_T - B_{T-s}$, we have that

$$\int_{T-t}^{T} H_s d\overleftarrow{B}_s = \int_0^t H'_s dB'_s.$$

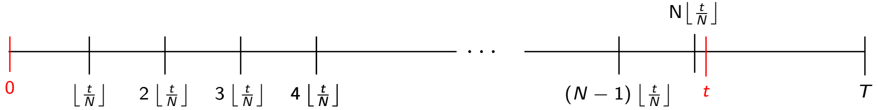
Consider two intervals of equal size t, $[0, t]$ and $[T-t, T]$, each partitioned into N subintervals:

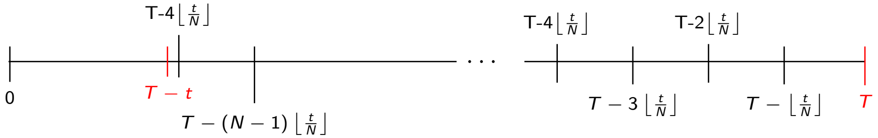




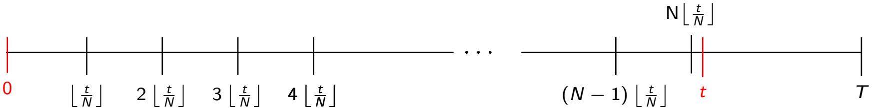


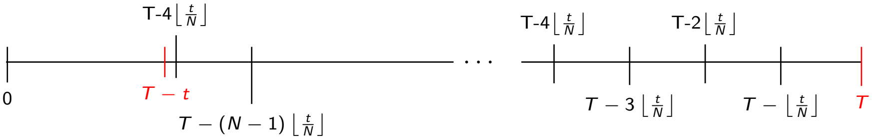
$$
\begin{aligned}
&\int_{T-t}^{T} H_s d\overleftarrow{B}_s \\
&= \lim_{N \nearrow \infty} \sum_{s \in \left\{T-(N-1)\left\lfloor \frac{t}{N} \right\rfloor, T-(N-2)\left\lfloor \frac{t}{N} \right\rfloor, \ldots, T-\left\lfloor \frac{t}{N} \right\rfloor, T\right\}} H_s \left(B_s - B_{s-\left\lfloor \frac{t}{N} \right\rfloor}\right) \\
&= \lim_{N \nearrow \infty} \Big\{ H_{T-(N-1)\left\lfloor \frac{t}{N} \right\rfloor} \left(B_{T-(N-1)\left\lfloor \frac{t}{N} \right\rfloor} - B_{T-N\left\lfloor \frac{t}{N} \right\rfloor}\right) \\
&\qquad + H_{T-(N-2)\left\lfloor \frac{t}{N} \right\rfloor} \left(B_{T-(N-2)\left\lfloor \frac{t}{N} \right\rfloor} - B_{T-(N-1)\left\lfloor \frac{t}{N} \right\rfloor}\right) \\
&\qquad + \ldots + H_{T-\left\lfloor \frac{t}{N} \right\rfloor} \left(B_{T-\left\lfloor \frac{t}{N} \right\rfloor} - B_{T-2\left\lfloor \frac{t}{N} \right\rfloor}\right) + H_T \left(B_T - B_{T-\left\lfloor \frac{t}{N} \right\rfloor}\right) \Big\}
\end{aligned}
$$





$$
\begin{aligned}
&\int_0^t H'_s \, dB'_s \\
&= \int_0^t H_{T-s} \, d(B_T - B_{T-s}) \\
&= \lim_{N \nearrow \infty} \sum_{s \in \left\{0,1,\dots,(N-2)\left\lfloor \frac{t}{N} \right\rfloor,(N-1)\left\lfloor \frac{t}{N} \right\rfloor\right\}} H_{T-s} \left(\left[B_T - B_{T-(s+\left\lfloor \frac{t}{N} \right\rfloor)} \right] - [B_T - B_{T-s}] \right) \\
&= \lim_{N \nearrow \infty} \sum_{s \in \left\{0,1,\dots,(N-2)\left\lfloor \frac{t}{N} \right\rfloor,(N-1)\left\lfloor \frac{t}{N} \right\rfloor\right\}} H_{T-s} \left(B_{T-s} - B_{T-(s+\left\lfloor \frac{t}{N} \right\rfloor)} \right)
\end{aligned}
$$





$$
\begin{aligned}
&\int_0^t H'_s \, dB'_s \\
&= \lim_{N \nearrow \infty} \sum_{s \in \{0,1,\dots,(N-2)\lfloor \frac{t}{N} \rfloor,(N-1)\lfloor \frac{t}{N} \rfloor\}} H_{T-s}\left(B_{T-s} - B_{T-(s+\lfloor \frac{t}{N} \rfloor)}\right) \\
&= \lim_{N \nearrow \infty} \Big\{ H_T\left(B_T - B_{T-\lfloor \frac{t}{N} \rfloor}\right) + H_{T-\lfloor \frac{t}{N} \rfloor}\left(B_{T-\lfloor \frac{t}{N} \rfloor} - B_{T-2\lfloor \frac{t}{N} \rfloor}\right) + \dots \\
&\qquad + H_{T-(N-2)\lfloor \frac{t}{N} \rfloor}\left(B_{T-(N-2)\lfloor \frac{t}{N} \rfloor} - B_{T-(N-1)\lfloor \frac{t}{N} \rfloor}\right) \\
&\qquad + H_{T-(N-1)\lfloor \frac{t}{N} \rfloor}\left(B_{T-(N-1)\lfloor \frac{t}{N} \rfloor} - B_{T-N\lfloor \frac{t}{N} \rfloor}\right) \Big\}
\end{aligned}
$$

Stochastic optimal control

$W \in \mathbb{R}^k$ is a $(\Omega, \mathcal{F}, \mathbb{P})$-Brownian motion,
$X \in \mathbb{R}^m$, $u \in \mathcal{U}$,
$b \in \mathcal{C}(\mathbb{R}^m \times \mathcal{U}, \mathbb{R}^m)$, $\sigma \in \mathcal{C}(\mathbb{R}^m \times \mathcal{U}, \mathbb{R}^m \times \mathbb{R}^k)$, σ is $\mathcal{F}_t$-measurable,
$(\sigma\sigma^*) \in \mathcal{C}(\mathbb{R}^m \times \mathcal{U}, \mathbb{R}^m \times \mathbb{R}^m)$.
$r \in \mathcal{C}(\mathbb{R}^m \times \mathcal{U}, \mathbb{R})$ and $g \in \mathcal{C}(\mathbb{R}^m, \mathbb{R})$ are both $\mathcal{F}_t$-measurable, r and g are both convex (concave)

$$\begin{aligned} \text{Signal:} \quad & dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dW_t \\ \text{Cost function:} \quad & \mathbb{E}\left[\mathrm{J}(u)\right] \\ \text{where} \quad & \mathrm{J}(u) := \int_0^T r(X_s, u_s)ds + g(X_T) \end{aligned}$$

Maximum Principle[2]

Consider scalar case: $X \in \mathbb{R}$, $W \in \mathbb{R}$.

Let

$$\tilde{\mathtt{J}}(u) := \int_0^T \{r(X_s, u_s)ds + p_s\left[dX_s - b(X_s, u_s)ds - \sigma(X_s, u_s)dW_s\right]\} + g(X_T)$$

Let u^o denote the optimal control and $0 < \varepsilon << 1$. Then,

$$0 = D\tilde{\mathtt{J}}(u^o) = \tilde{\mathtt{J}}(u^o + \varepsilon u) - \tilde{\mathtt{J}}(u^o)$$

Let (X^o, p^o) be the signal and Lagrange multiplier under control u^o.
Let $(X^\varepsilon, p^\varepsilon)$ be the signal and Lagrange multiplier under control $u^\varepsilon := u^o + \varepsilon u$

[2]Bismut, Conjugate Convex Functions in Optimal Stochastic Control, J. Math. Analysis and Appl. (1973)

$$
\begin{aligned}
&\tilde{J}(u^o + \varepsilon u) \\
&= \int_0^T \{ r(X_s^\varepsilon, u_s^\varepsilon) ds + p_s^\varepsilon \left[dX_s^\varepsilon - b(X_s^\varepsilon, u_s^\varepsilon) ds - \sigma(X_s^\varepsilon, u_s^\varepsilon) dW_s \right] \} + g(X_T^\varepsilon) \\
&= \int_0^T \Bigg\{ r(X_s^\varepsilon, u_s^\varepsilon) ds \\
&\qquad + p_s^\varepsilon \left[dX_s^\varepsilon - \left(b(X_s^\varepsilon, u_s^\varepsilon) - \frac{1}{2} \sigma(X_s^\varepsilon, u_s^\varepsilon) \sigma_x(X_s^\varepsilon, u_s^\varepsilon) \right) ds - \sigma(X_s^\varepsilon, u_s^\varepsilon) \circ dW_s \right] \Bigg\} \\
&\quad + g(X_T^\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{J}(u^o + \varepsilon u) - \tilde{J}(u^o) \\
&= \int_0^T \Big\{ (r_x(X_s^o, u_s^o))^* (X_s^\varepsilon - X_s^o) ds \\
&\qquad - p_s^o \Big[\Big(b_x(X_s^o, u_s^o) - \frac{1}{2} \left(\sigma(X_s^o, u_s^o) \sigma_x(X_s^o, u_s^o) \right)_x \Big) (X_s^\varepsilon - X_s^o) ds \\
&\qquad\qquad + \sigma_x(X_s^o, u_s^o)(X_s^\varepsilon - X_s^o) \circ dW_s \Big] \Big\} \\
&\quad + \int_0^T p_s^o \, d(X_s^\varepsilon - X_s^o) \\
&\quad + \int_0^T \Big\{ r_u(X_s^o, u_s^o)(\varepsilon u_s) ds \\
&\qquad - p_s^o \Big[\Big(b_u(X_s^o, u_s^o) - \frac{1}{2} \left(\sigma(X_s^o, u_s^o) \sigma_x(X_s^o, u_s^o) \right)_u \Big) (\varepsilon u_s) ds \\
&\qquad\qquad + \sigma_u(X_s^o, u_s^o)(\varepsilon u_s) \circ dW_s \Big] \Big\} \\
&\quad - \int_0^T (p_s^\varepsilon - p_s^o) \left(dx_s^o - \left(b(x_s^o, u_s^o) - \frac{1}{2} \sigma(X_s^o, u_s^o) \sigma_x(X_s^o, u_s^o) \right) ds - \sigma(x_s^o, u_s^o) \circ dW_s \right) \\
&\quad + g_x(X_T^o)(X_T^\varepsilon - X_T^o)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{J}(u^o + \varepsilon u) - \tilde{J}(u^o) \\
&= \int_0^T \Big\{ (r_x(X_s^o, u_s^o))^* (X_s^\varepsilon - X_s^o) ds \\
&\qquad -p_s^o \Big[\Big(b_x(X_s^o, u_s^o) - \frac{1}{2} \left(\sigma(X_s^o, u_s^o)\sigma_x(X_s^o, u_s^o)\right)_x \Big) (X_s^\varepsilon - X_s^o) ds \\
&\qquad\qquad +\sigma_x(X_s^o, u_s^o)(X_s^\varepsilon - X_s^o) \circ dW_s \Big] \Big\} \\
&\quad + p_T^o (X_T^\varepsilon + X_T^o) - p_0^o (X_0^\varepsilon - X_0^o) - \int_0^T dp_s^o \, (X_s^\varepsilon - X_s^o) \\
&\quad + \int_0^T \Big\{ r_u(X_s^o, u_s^o)(\varepsilon u_s) ds \\
&\qquad -p_s^o \Big[\Big(b_u(X_s^o, u_s^o) - \frac{1}{2} \left(\sigma(X_s^o, u_s^o)\sigma_x(X_s^o, u_s^o)\right)_u \Big) (\varepsilon u_s) ds \\
&\qquad\qquad +\sigma_u(X_s^o, u_s^o)(\varepsilon u_s) \circ dW_s \Big] \Big\} \\
&\quad - \int_0^T (p_s^\varepsilon - p_s^o) \Big(dx_s^o - \Big(b(x_s^o, u_s^o) - \frac{1}{2}\sigma(X_s^o, u_s^o)\sigma_x(X_s^o, u_s^o) \Big) ds - \sigma(x_s^o, u_s^o) \circ dW_s \Big) \\
&\quad + g_x(X_T^o)(X_T^\varepsilon - X_T^o)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{J}(u^o + \varepsilon u) - \tilde{J}(u^o) \\
&= \int_0^T \Big\{ (r_x(X_s^o, u_s^o))^*(X_s^\varepsilon - X_s^o) ds \\
&\qquad - p_s^o \Big[\Big(b_x(X_s^o, u_s^o) - \frac{1}{2} (\sigma(X_s^o, u_s^o)\sigma_x(X_s^o, u_s^o))_x \Big) (X_s^\varepsilon - X_s^o) ds \\
&\qquad\qquad + \sigma_x(X_s^o, u_s^o)(X_s^\varepsilon - X_s^o) \circ dW_s \Big] \Big\} \\
&\quad + p_T^o(X_T^\varepsilon + X_T^o) - p_0^o(X_0^\varepsilon - X_0^o) - \int_0^T dp_s^o \, (X_s^\varepsilon - X_s^o) \\
&\quad + \int_0^T \Big\{ r_u(X_s^o, u_s^o)(\varepsilon u_s) ds \\
&\qquad - p_s^o \Big[\Big(b_u(X_s^o, u_s^o) - \frac{1}{2} (\sigma(X_s^o, u_s^o)\sigma_x(X_s^o, u_s^o))_u \Big) (\varepsilon u_s) ds \\
&\qquad\qquad + \sigma_u(X_s^o, u_s^o)(\varepsilon u_s) \circ dW_s \Big] \Big\} \\
&\quad - \int_0^T (p_s^\varepsilon - p_s^o) \Big(dx_s^o - \Big(b(x_s^o, u_s^o) - \frac{1}{2}\sigma(X_s^o, u_s^o)\sigma_x(X_s^o, u_s^o) \Big) ds - \sigma(x_s^o, u_s^o) \circ dW_s \Big) \\
&\quad + g_x(X_T^o)(X_T^\varepsilon - X_T^o)
\end{aligned}
$$

$$
\begin{aligned}
dX_s^\varepsilon &= d(X_s^\varepsilon - X_s^o + X_s^o) \\
&= dX_s^o + d(X_s^\varepsilon - X_s^o)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T p_s^o \, d(X_s^\varepsilon - X_s^o) \\
&= p_T^o(X_T^\varepsilon - X_T^o) - p_0^o(X_0^\varepsilon - X_0^o) - \int_0^T dp_s^o \, (X_s^\varepsilon - X_s^o) \\
&= p_T^o(X_T^\varepsilon - X_T^o) - \int_0^T dp_s^o \, (X_s^\varepsilon - X_s^o) \qquad (\text{since } X_0^\varepsilon = X_0^o = X_0)
\end{aligned}
$$

Stratonovich equations

Let $H(t,x,u,p) :=$
$\int_0^t \left\{-r(x_s,u_s)ds + p_s \left[\left(b(x_s,u_s) - \frac{1}{2}\sigma(x_s,u_s)\sigma_x(x_s,u_s)\right) ds + \sigma(x_s,u_s)\circ dW_s\right]\right\}$,

- $H_u(t, X^o, u^o, p^o) = 0$
- Forward state SDE:

$$dX_t^o = \left(b(X_t^o,u_t^o) - \frac{1}{2}\sigma(X_t^o,u_t^o)\sigma_x(X_t^o,u_t^o)\right) dt + \sigma(X_t^o,u_t^o)\circ dW_t$$
$$X_0^o = X_0$$

- Backward costate SDE:

$$dp_t^o = r_x(X_t^o,u_t^o)dt - p_t^o\left(b(X_t^o,u_t^o) - \frac{1}{2}\sigma(X_t^o,u_t^o)\sigma_x(X_t^o,u_t^o)\right)_x dt$$
$$- p_t^o\,\sigma_x(X_t^o,u_t^o)\circ dW_t,$$
$$p_T^o = -g_x(X_T^o)$$

Stratonovich equations

Let $H(t, x, u, p) :=$
$\int_0^t \left\{-r(x_s, u_s)ds + p_s \left[\left(b(x_s, u_s) - \frac{1}{2}\sigma(x_s, u_s)\sigma_x(x_s, u_s)\right) ds + \sigma(x_s, u_s) \circ dW_s\right]\right\}$

- $H_u(t, X^o, u^o, p^o) = 0$
- Forward state SDE:

$$dX_t^o = dH_p(t, X_t^o, u_t^o, p_t^o),$$
$$X_0^o = X_0$$

- Backward costate SDE:

$$dp_t^o = -dH_x(t, X_t^o, u_t^o, p_t^o),$$
$$p_T^o = g_x(X_T^o)$$

Itô equations

Let $H(t, x, u, p) :=$
$\int_0^t \left\{-r(x_s, u_s)ds + p_s \left[\left(b(x_s, u_s) - \frac{1}{2}\sigma(x_s, u_s)\sigma_x(x_s, u_s)\right) ds + \sigma(x_s, u_s) \circ dW_s\right]\right\}$,

- $H_u(t, X^o, u^o, p^o) = 0$
- Forward state SDE:

$$\begin{aligned} dX_t^o &= b(X_t^o, u_t^o)dt + \sigma(X_t^o, u_t^o)dW_t \\ X_0^o &= X_0 \end{aligned}$$

- Backward costate SDE:

$$\begin{aligned} dp_t^o &= r_x(X_t^o, u_t^o)dt - p_t^o \left(b(X_t^o, u_t^o) - \frac{1}{2}\sigma(X_t^o, u_t^o)\sigma_x(X_t^o, u_t^o)\right)_x dt \\ &\quad - \frac{1}{2}\left\langle d\left(p^o\sigma_x(X^o, u^o), W\right)\right\rangle_t - p_t^o\,\sigma_x(X_t^o, u_t^o)dW_t, \\ p_T^o &= -g_x(X_T^o) \end{aligned}$$

Itô equations

$$d\left(p_t^o \sigma_x(X_t^o, u_t^o)\right) = \left[\left(p_t^o\right)_x \sigma_x(X_t^o, u_t^o) + p_t^o \sigma_{xx}(X_t^o, u_t^o)\right] \sigma(X_t^o, u_t^o) dW_t + (\ldots) dt$$

Backward costate SDE:

$$\begin{aligned}
dp_t^o &= r_x(X_t^o, u_t^o)dt - \left[p_t^o b_x(X_t^o, u_t^o) + \frac{1}{2} p_t^o \sigma_x(X_t^o, u_t^o)^2 + \frac{1}{2} p_t^o \sigma_{xx}(X_t^o, u_t^o)\sigma(X_t^o, u_t^o)\right] dt \\
&\quad - \frac{1}{2}\left[\left(p_t^o\right)_x \sigma_x(X_t^o, u_t^o)\sigma(X_t^o, u_t^o) - \frac{1}{2} p_t^o \sigma_{xx}(X_t^o, u_t^o)\sigma(X_t^o, u_t^o)\right] dt - p_t^o \sigma_x(X_t^o, u_t^o) dW_t \\
&= r_x(X_t^o, u_t^o)dt - p_t^o b_x(X_t^o, u_t^o)dt - \frac{1}{2}\left[\left(p_t^o\right)_x \sigma(X_t^o, u_t^o) - p_t^o \sigma_x(X_t^o, u_t^o)\right] \sigma_x(X_t^o, u_t^o) dt \\
&\quad - p_t^o \sigma_x(X_t^o, u_t^o) dW_t \\
&\ \ \vdots \\
dp_t^o &= \left[r_x(X_t^o, u_t^o) - p_t^o b_x(X_t^o, u_t^o) - \left(p_t^o\right)_x \sigma_x(X_t^o, u_t^o)\sigma(X_t^o, u_t^o)\right] dt + (p_t^o)_x \sigma(X_t^o, u_t^o) dW_t, \\
p_T^o &= -g_x(X_T^o)
\end{aligned}$$

Dynamic programming

Return to $X \in \mathbb{R}^m$, $W \in \mathbb{R}^k$. Let

$$V(t,x) := \inf_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^T r(X_s,u_s)ds + g(X_T)\right]$$
$$= \inf_{u\in\mathcal{U}} \mathbb{E}\left[\int_t^T r(X_s,u_s)ds + g(X_T)\bigg| X_t = x\right], \qquad V(T,x) = g(x)$$

Maximality principle: For $0 < h < T - t$,

$$V(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^{t+h} r(X_s,u_s)ds + V(t+h, X_{t+h})\right]$$
$$0 = \inf_{u\in\mathcal{U}} \mathbb{E}_{t,x}\left[\int_t^{t+h} r(X_s,u_s)ds + V(t+h, X_{t+h} - V(t,x))\right]$$
$$0 = \inf_{u\in\mathcal{U}} \lim_{h\searrow 0} \mathbb{E}_{t,x}\left[\frac{1}{h}\int_t^{t+h} r(X_s,u_s)ds + \frac{1}{h}\left(V(t+h, X_{t+h} - V(t,x))\right)\right]$$
$$0 = \inf_{u\in\mathcal{U}} \left\{ r(x,u) + \lim_{h\searrow 0}\frac{1}{h}\mathbb{E}_{t,x}\left[V(t+h, X_{t+h} - V(t,x)\right]\right\}$$
$$0 = \inf_{u\in\mathcal{U}} \left\{ r(x,u) + \frac{\partial}{\partial t}V(t,x) + \mathcal{L}V(t,x)\right\}$$

Dynamic programming

$$\frac{\partial}{\partial t}V(t,x) + \mathcal{L}^o V(t,x) + r(x, u_t^o) = 0,$$

$\mathcal{L}^o$ is generator of X under optimal u^o.

We would like to find a SDE representation of the solution to the PDE.
Let $Y_s := V(s, (X_s^o)^{x,t})$ where

$$d(X_s^o)^{x,t} = b((X_s^o)^{x,t}, u_t)dt + \sigma((X_s^o)^{x,t}, u_t)dW_t, \qquad \text{for s} \in (\text{t}, \text{T}],$$
$$(X_t^o)^{x,t} = x \qquad \text{for s} \leq \text{t}.$$

By Itô's lemma,

$$\begin{aligned} dY_s &= \left\{ \frac{\partial}{\partial t}V(s,(X_s^o)^{x,t}) + \mathcal{L}^o V(s,(X_s^o)^{x,t}) \right\} ds \\ &\quad + \left(\nabla_x V(s,(X_s^o)^{x,t})\right)^* \sigma((X_s^o)^{x,t}, u_s^o)dW_s \\ &= -r((X_s^o)^{x,t}, u_s^o)ds + (\nabla_x Y_s)^* \sigma((X_s^o)^{x,t}, u_s^o)dW_s, \\ Y_T &= V(T,(X_T^o)^{x,t}) = g((X_T^o)^{x,t}) \end{aligned}$$

Relation between maximum principle and dynamic programming

Let $p_t^o = p(t, X_t^o)$. Then,

$$dp(t, X_t^o) = \left[\frac{\partial p(t, X_t^o)}{\partial t} + \mathcal{L}p(t, X_t^o)\right] dt + p_x(t, X_t^o)\sigma(X_t^o, u_t^o)dW_t$$

Backward costate equation:

$$\begin{aligned}
&\left[\frac{\partial p(t, x)}{\partial t} + \mathcal{L}p(t, x)\right] dt + p_x(t, x)\sigma(x, u)dW_t \\
&= [r_x(x, u) - p(t, x)b_x(x, u) - p_x(t, x)\sigma_x(x, u)\sigma(x, u)]\, dt + p_x(t, x)\sigma(x, u)dW_t, \\
&p(T, x) = -g_x(x)
\end{aligned}$$

Relation between maximum principle and dynamic programming

HJB equation:

$$\frac{\partial V(t,x)}{\partial t} + \mathcal{L}V(t,x) = -r(x,u)$$

$$\frac{\partial V_x(t,x)}{\partial t} + \mathcal{L}V_x(t,x) = -r_x(x,u) - V_x(t,x)b_x(x,u) - V_{xx}(t,x)\sigma_x(x,u)\sigma(x,u),$$

$$V_x(T,x) = g_x(x)$$

Relation between maximum principle and dynamic programming

HJB equation:

$$\frac{\partial V(t,x)}{\partial t} + \mathcal{L}V(t,x) = -r(x,u)$$

$$\frac{\partial V_x(t,x)}{\partial t} + \mathcal{L}V_x(t,x) = -r_x(x,u) - V_x(t,x)b_x(x,u) - V_{xx}(t,x)\sigma_x(x,u)\sigma(x,u),$$

$$V_x(T,x) = g_x(x)$$

Let $\frac{\partial V(t,x)}{\partial x} = -p(t,x)$. Then we get the relation from the backward costate equation (time integral part):

$$\frac{\partial p(t,x)}{\partial t} + \mathcal{L}p(t,x) = r_x(x,u) - p(t,x)b_x(x,u) - p_x(t,x)\sigma_x(x,u)\sigma(x,u),$$

$$p(T,x) = -g_x(x)$$

Existence and uniqueness of solution to the BSDE[3][4]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ support a k-dimensional Brownian motion W.
Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the completed filtration generated by W.

- $\mathcal{P}_n$ is set of $\mathcal{F}_t$-progressively measurable process $\varphi : [0, T] \times \Omega \to \mathbb{R}^n$
- $L^2_n(\mathcal{F}_t) := \{\zeta : \ \zeta \text{ is } \mathcal{F}_t\text{-measurable}, \zeta \in \mathbb{R}^m, \mathbb{E}[|\zeta|^2] < \infty\}$
- $\mathcal{S}^2_n(0, T) := \{\varphi \in \mathcal{P}_n : \ \varphi \text{ has continuous paths}, \mathbb{E}[\sup_{t\leq T} |\varphi_t|^2] < \infty\}$
- $\mathcal{H}^p_n(0, T) := \left\{ Z \in \mathcal{P}_n : \mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds \right)^{1/p} \right] < \infty \right\}$

Let $\zeta_T \in L^2_n(\mathcal{F}_T)$ be a terminal condition and $f : \mathcal{P}_m \times \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^{n\times k}) \to \mathbb{R}^n$.
A solution to the BSDE with parameters (f, ζ_T) is a pair of $\mathcal{F}_t$-progressively measurable processes $(Y_t, Z_t) \in \mathbb{R}^n \times \mathbb{R}^{n\times k}$ if

$$Y \in \mathcal{S}^2_n(0, T), \qquad Z \in \mathcal{H}^2_{n\times k}(0, T),$$
$$dY_t = -f(t, \omega, Y_t, Z_t)dt + Z_t dW_t, \qquad Y_T = \zeta_T.$$

[3] El Karoui, Hamadéne and Matoussi, Backward Stochastic Differential Equations and Applications (2008)

[4] Pardoux and Peng, Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations (1992)

Assumptions:

- $f(t, \omega, 0, 0) \in \mathcal{H}_n^2$
- f is uniformly Lipschitz in (y, z): $\exists$ constant $K \geq 0$ s.t. $\forall (y, y', z, z')$,

$$|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq K(|y - y'| + |z - z'|) \quad dt \otimes d\mathbb{P} \text{ a.e.}$$

Case $f \equiv 0$: $Y_t = \zeta_T - \int_t^T Z_s dW_s$

Taking conditional expectation w.r.t. $\mathcal{F}_t$:

$$\underbrace{\mathbb{E}[Y_t|\mathcal{F}_t]}_{\substack{= Y_t \text{ because } Y_t \text{ is} \\ \mathcal{F}_t\text{-measurable}}} = \underbrace{\mathbb{E}[\zeta_T|\mathcal{F}_T]}_{= \mathbb{E}[Y_T|\mathcal{F}_t]} - \underbrace{\mathbb{E}\left[\int_t^T Z_s dW_s \middle| \mathcal{F}_t\right]}_{\substack{= \mathbb{E}\left[\int_t^T Z_s dW_s\right] \text{ because} \\ (W_s - W_t,\, s \in [t, T]) \text{ independent of } \mathcal{F}_t}}$$

$$Y_t = \mathbb{E}[Y_T|\mathcal{F}_t] \implies Y_t \text{ is an } \mathcal{F}_t\text{-martingale}$$

Case $f \equiv 0$: $Y_t = \zeta_T - \int_t^T Z_s dW_s$
By martingale representation theorem[5],
(**existence**) $\exists$ Z, $\mathbb{E}\left[\int_0^T Z_s^2 ds\right] < \infty$ s.t.

$$\begin{aligned} Y_t &= Y_0 + \int_0^t Z_s dW_s \\ &= Y_r + \int_r^t Z_s dW_s \ \forall t \in [0, T] \implies Y_r = Y_T - \int_r^T Z_s dW_s \end{aligned}$$

(**uniqueness**) In addition, if $\exists$ $\tilde{Z}$, $\mathbb{E}\left[\int_0^T \tilde{Z}_s^2 ds\right] < \infty$ s.t. $Y_t = Y_0 + \int_0^t \tilde{Z}_s dW_s$, then

$$\int_0^\infty |Z_s - \tilde{Z}_s|^2 ds = 0 \quad \text{a.s.} \quad \left(\mathbb{P}\left[\lim_{t \nearrow \infty} \int_0^t |Z_s - \tilde{Z}_s|^2 ds\right] = 0\right)$$

[5] Thm. 3.4.15, Karatzas and Shreve, Brownian Motion and Stochastic Calculus

Case $f = f(t, \omega)$, independent of (y, z): $Y_t = \zeta_T + \int_t^T f(s)ds - \int_t^T Z_s dW_s$

Let $\tilde{Y}_t := Y_t + \int_0^t f(s)ds$, $\tilde{Y}_T = \zeta_T + \int_0^T f(s)ds$. Then

$$\begin{aligned}\tilde{Y}_T &= Y_t - \int_t^T f(s)ds + \int_t^T Z_s dW_s + \int_0^T f(s)ds \\ &= Y_t + \int_0^t f(s)ds + \int_t^T Z_s dW_s = \tilde{Y}_t + \int_t^T Z_s dW_s\end{aligned}$$

Martingale representation theorem gives existence and uniqueness of Z, $\mathbb{E}\left[\int_0^T Z_s^2 ds\right] < \infty$ s.t. $\tilde{Y}_T = \tilde{Y}_t + \int_t^T Z_s dW_s$. Then

$$\tilde{Y}_T = \tilde{Y}_t + \int_t^T Z_s dW_s$$

$$\zeta_T + \int_0^T f(s)ds = Y_t + \int_0^t f(s)ds + \int_t^T Z_s dW_s$$

$$Y_t = \zeta_T + \int_t^T f(s)ds - \int_t^T Z_s dW_s$$

Case $f = f(t, \omega, y, z)$: $Y_t = \zeta_T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$

Let $(Y_t^{u,v}, Z_t^{u,v})$ is the solution to the BSDE

$$dY_t^{u,v} = -f(t, u_t, v_t)dt + Z_t^{u,v} dW_t, \qquad Y_T^{u,v} = \zeta_T,$$

(u, v) independent of (Y, Z).
Solution to above BSDE exists and is unique from the previous case, $f = f(t, \omega)$.

Define a map $\Phi : \mathcal{H}^\alpha \to \mathcal{H}^\alpha$:

$$(u_t, v_t)_{t \in [0,T]} \in \mathcal{H}^\alpha, \qquad \Phi(u, v) = (Y_t^{u,v}, Z_t^{u,v})_{t \in [0,T]},$$

Define a norm on $\mathcal{H}^\alpha := \mathcal{H}_n^2 \times \mathcal{H}_{n\times k}^2$, $\alpha > 0$:

$$\|(Y, Z)\|_\alpha := \left(\mathbb{E}\left[\int_0^T e^{\alpha s}(|Y_s|^2 + |Z_s|^2)\, ds \right] \right)^{1/2}$$

Apply Itô's lemma to $e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2$:

$$
\begin{aligned}
& d\left[e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2\right] \\
& = \alpha e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2 dt + 2e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})(dY_t^{u,v} - dY_t^{u',v'}) \\
& \quad + e^{\alpha t}(d\langle Y^{u,v}\rangle_t + d\langle Y^{u',v'}\rangle_t - 2d\langle Y_t^{u,v}, Y_t^{u',v'}\rangle) \\
& = \alpha e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2 dt \\
& \quad + 2e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})(-[f(t,u,v) - f(t,u',v')]dt + [Z_t^{u,v} - Z_t^{u',v'}]dW_t) \\
& \quad + e^{\alpha t}(Z_t^{u,v} - Z_t^{u',v'})^2 dt
\end{aligned}
$$

Integrating from t to T:

$$\begin{aligned} & e^{\alpha T}(Y_T^{u,v} - Y_T^{u',v'})^2 - e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2 \\ &= \alpha \int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})^2 ds \\ &\quad - 2\int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})(f(s,u,v) - f(s,u',v'))ds \\ &\quad + 2\int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})(Z_s^{u,v} - Z_s^{u',v'})dW_s \\ &\quad + \int_t^T e^{\alpha s}(Z_s^{u,v} - Z_s^{u',v'})^2 ds \end{aligned}$$

Integrating from t to T:
Rearranging and taking expectation,

$$\mathbb{E}\left[e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2\right] + \mathbb{E}\left[\int_t^T e^{\alpha s}(Z_s^{u,v} - Z_s^{u',v'})^2 ds\right]$$

$$= e^{\alpha T} \underbrace{\mathbb{E}\left[(Y_T^{u,v} - Y_T^{u',v'})^2\right]}_{=\mathbb{E}[\zeta_T - \zeta_T] = 0}$$

$$- \alpha\mathbb{E}\left[\int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})^2 ds\right]$$

$$+ 2\mathbb{E}\left[\int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})(f(s,u,v) - f(s,u',v'))ds\right]$$

$$- 2\mathbb{E}\left[\int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})(Z_s^{u,v} - Z_s^{u',v'})dW_s\right]$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_t^T e^{2\alpha s}(Y_s^{u,v} - Y_s^{u',v'})^2 (Z_s^{u,v} - Z_s^{u',v'})^2 ds\right)^{1/2}\right] \\
&\leq \mathbb{E}\left[\left(\int_t^T C_1 \left\{\sup_{t\leq s\leq T}(Y_s^{u,v} - Y_s^{u',v'})^2\right\}(Z_s^{u,v} - Z_s^{u',v'})^2 ds\right)^{1/2}\right] \\
&= C_2\mathbb{E}\left[\left(\left\{\sup_{t\leq s\leq T}(Y_s^{u,v} - Y_s^{u',v'})^2\right\}\int_t^T (Z_s^{u,v} - Z_s^{u',v'})^2 ds\right)^{1/2}\right] \\
&= \frac{C_2}{\sqrt{2}}\mathbb{E}\left[\sup_{t\leq s\leq T}\underbrace{\left(Y_s^{u,v} - Y_s^{u',v'}\right)}_{\in\mathcal{S}_n^2}^2 + \int_t^T \underbrace{\left(Z_s^{u,v} - Z_s^{u',v'}\right)}_{\in\mathcal{H}_{n\times k}^2}^2 ds\right] \\
&< \infty
\end{aligned}
$$

So, $\int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})(Z_s^{u,v} - Z_s^{u',v'})dW_s$ is a square-integrable martingale and expected value is zero

By Lipschitz assumption on f, $\exists\, K \geq 0$ s.t.

$$\mathbb{E}\left[\int_t^T e^{\alpha s}\left\{-\alpha(Y_s^{u,v} - Y_s^{u',v'})^2 + 2(Y_s^{u,v} - Y_s^{u',v'})(f(s,u,v) - f(s,u',v'))\right\} ds\right]$$

$$\leq \mathbb{E}\left[\int_t^T e^{\alpha s}\left\{-\alpha(Y_s^{u,v} - Y_s^{u',v'})^2 + 2K(Y_s^{u,v} - Y_s^{u',v'})(|u-u'| + |v-v'|)\right\} ds\right]$$

$$
\begin{aligned}
-\alpha a^2 + 2Kab &= -\alpha \left(a^2 + 2\frac{K}{\alpha} ab + \frac{K^2}{\alpha^2} b^2 - \frac{K^2}{\alpha^2} b^2 \right) \\
&= -\alpha \left(a^2 + \frac{K}{\alpha} b \right)^2 + \frac{K^2}{\alpha^2} b^2
\end{aligned}
$$

so

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^T e^{\alpha s} \left\{ -\alpha (Y_s^{u,v} - Y_s^{u',v'})^2 + 2(Y_s^{u,v} - Y_s^{u',v'})(f(s,u,v) - f(s,u',v')) \right\} ds \right] \\
&\leq \mathbb{E}\left[\int_t^T e^{\alpha s} \left\{ -\alpha (Y_s^{u,v} - Y_s^{u',v'})^2 + 2K(Y_s^{u,v} - Y_s^{u',v'})(|u-u'| + |v-v'|) \right\} ds \right] \\
&= \mathbb{E}\left[\int_t^T e^{\alpha s} \left\{ -\alpha \left[(Y_s^{u,v} - Y_s^{u',v'}) + \frac{K}{\alpha}(|u-u'| + |v-v'|) \right]^2 \right.\right. \\
&\qquad\qquad \left.\left. + \frac{K^2}{\alpha}(|u-u'| + |v-v'|)^2 \right\} ds \right] \\
&\leq \frac{K^2}{\alpha} \mathbb{E}\left[\int_t^T e^{\alpha s} (|u-u'| + |v-v'|)^2 ds \right]
\end{aligned}
$$

Back to full integrated equation:

$$\mathbb{E}\left[e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2\right] + \mathbb{E}\left[\int_t^T e^{\alpha s}(Z_s^{u,v} - Z_s^{u',v'})^2 ds\right]$$
$$\leq \frac{K^2}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}(|u - u'| + |v - v'|)^2 ds\right]$$
$$\leq \frac{2K^2}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}(|u - u'|^2 + |v - v'|^2) ds\right]$$

Back to full integrated equation:

$$\mathbb{E}\left[e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2\right] + \mathbb{E}\left[\int_t^T e^{\alpha s}(Z_s^{u,v} - Z_s^{u',v'})^2 ds\right]$$
$$\leq \frac{2K^2}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}(|u-u'|^2 + |v-v'|^2)ds\right]$$

Let $\beta \in [0,1]$:

$$\mathbb{E}\left[\int_t^T e^{\alpha s}(Z_s^{u,v} - Z_s^{u',v'})^2 ds\right] \leq \frac{2\beta K^2}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}(|u-u'|^2 + |v-v'|^2)ds\right]$$

and

$$\mathbb{E}\left[e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2\right] \leq \frac{2(1-\beta)K^2}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}(|u-u'|^2 + |v-v'|^2)ds\right]$$

$$\mathbb{E}\left[\int_t^T e^{\alpha s}(Y_s^{u,v} - Y_s^{u',v'})^2 ds\right] \leq \frac{2(1-\beta)K^2}{\alpha}\left(\frac{1}{\alpha}[e^{\alpha T} - e^{\alpha t}]\right)$$
$$\times\,\mathbb{E}\left[\int_t^T e^{\alpha s}(|u-u'|^2 + |v-v'|^2)ds\right]$$

Back to full integrated equation:

$$\mathbb{E}\left[e^{\alpha t}(Y_t^{u,v} - Y_t^{u',v'})^2\right] + \mathbb{E}\left[\int_t^T e^{\alpha s}(Z_s^{u,v} - Z_s^{u',v'})^2 ds\right]$$
$$\leq \frac{2K^2}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}(|u-u'|^2 + |v-v'|^2)ds\right]$$

So,

$$\mathbb{E}\left[\int_t^T e^{\alpha s}\left\{(Y_t^{u,v} - Y_t^{u',v'})^2 + (Z_s^{u,v} - Z_s^{u',v'})^2\right\} ds\right]$$
$$\leq C(\alpha, K, T)\,\mathbb{E}\left[\int_t^T e^{\alpha s}(|u-u'|^2 + |v-v'|^2)ds\right]$$
$$\implies \|(Y,Z)\|_\alpha \leq C(\alpha, K, T)\,\mathbb{E}\left[\int_t^T e^{\alpha s}(|u-u'|^2 + |v-v'|^2)ds\right]$$

Fixed point theorem ensures there is a unique pair (Y,Z) s.t. $\Phi(Y,Z) = (Y,Z)$

Filtering

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space that supports a $k + d$-dimensional Brownian motion (W, B), W and B are independent.

$$\text{Signal}: \ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = \xi \in \mathbb{R}^m,$$
$$\text{Observation}: \ dY_t = h(X_t)dt + dB_t, \quad Y_0 = 0_{d\times 1} \in \mathbb{R}^d$$

$b \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{d\times k}$, $(\sigma\sigma^*) \in \mathcal{C}^2(\mathbb{R}^m, \mathbb{R}^m)$ $h : \mathbb{R}^m \to \mathbb{R}^d$.

Let $\{\mathcal{Y}_t\}_{t\geq 0}$ be the filtration generated by $(Y_t)_{t\geq 0}$. For $\mathcal{C}_b^2$ function φ,

$$\text{Filter}: \ \pi_t(\varphi) := \mathbb{E}_{\mathbb{Q}}\left[\varphi(X_t)\middle|\, \mathcal{Y}_t\right]$$

Brownian motion[a]

[a] Ch. 2, Øksendal, Stochastic Differential Equations, 5^{th} ed.

B is a $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$-Brownian motion if

- B_t is continuous a.s., $B_0 = 0$
- B_t is a zero-mean Gaussian process
- B_t has independent increments, $\mathbb{E}\left[(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] = \delta_{ij}(t_i - t_{i-1})$

Lévy's characterization of Brownian motion[a]

[a] App. B, Bain & Crisan, Fundamentals of Stochastic Filtering, 2009

Let $\mathcal{F}_t$ be the (completion of) filtration generated by $\{B_t^i\}_{i=1}^n$. Let B_t^i be a continuous local $\mathcal{F}_t$-martingale starting from zero for $i = 1, \ldots, n$.
$B_t = (B_t^1, B_t^2, \ldots, B_t^n)$ is an n-dimensional $(\Omega, \mathcal{F}, \mathbb{P})$-Brownian motion adapted to $\mathcal{F}_t$ if and only if

$$\left\langle B^i, B^j \right\rangle_t = \delta_{ij} t \quad \forall\, i, j \in \{1, \ldots, n\}.$$

Girsanov's Theorem[6]

Probability space: $(\Omega, \mathcal{F}, \mathbb{Q})$. Let M be a continuous $\mathcal{F}_\cdot$-martingale and

$$D_t := \exp\left\{\mathrm{M_t} - \frac{1}{2}\langle \mathrm{M} \rangle_\mathrm{t}\right\}.$$

If D is a uniformly integrable martingale, then a new measure $\mathbb{P}$, continuous w.r.t. $\mathbb{Q}$ can be defined by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = D_\infty.$$

In addition, if X is a continuous $\mathcal{F}_\cdot$-martingale under $\mathbb{Q}$, then $X_t - \langle X, M \rangle_t$ is a continuous $\mathcal{F}_\cdot$-martingale under $\mathbb{P}$ as well.

[6] App. B, Bain and Crisan, Stochastic Filtering Theory (2009)

Girsanov's Theorem[7]

Application to filtering: A new measure $\mathbb{P}$ can be defined by

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = D_t, \qquad D_t := \exp\left\{-\int_0^t h(X_s)^* dB_s - \frac{1}{2}\int_0^t \|h(X_s)\|^2 ds\right\}.$$

Can check that $\exp\left\{-\int_0^t h(X_s)^* dB_s - \frac{1}{2}\int_0^t \|h(X_s)\|^2 ds\right\}$ is a uniformly integrable martingale.

B is a $\mathbb{Q}$-Brownian motion, so it is a continuous $\mathcal{F}_\cdot$-martingale under $\mathbb{Q}$. Then

$$\tilde{B}_t := B_t - \left\langle B, -\int_0^\cdot h(X_s)^* dB_s \right\rangle_t = B_t + \int_0^t h(X_s) ds$$

is a continuous $\mathcal{F}_\cdot$-martingale under $\mathbb{P}$. Also,

$$\langle \tilde{B} \rangle_t = \left\langle B + \int_0^\cdot h(X_s) ds \right\rangle_t = \langle B \rangle_t = t,$$

so $\tilde{B}$ is a $\mathbb{P}$-Brownian motion by Lévy's characterization of Brownian motion ▸.

[7] App. B, Bain and Crisan, Stochastic Filtering Theory (2009)

Girsanov's Theorem[8]

Application to filtering: A new measure $\mathbb{P}$ can be defined by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = D_\infty, \qquad D_t := \exp\left\{-\int_0^t h(X_s)^* dB_s - \frac{1}{2}\int_0^t \|h(X_s)\|^2 ds\right\}.$$

Specifically, under $\mathbb{P}$,

$$Y_t = B_t + \int_0^t h(X_s)ds = \tilde{B}_t,$$

is a Brownian motion, independent of W that drives the signal.

[8] App. B, Bain and Crisan, Stochastic Filtering Theory (2009)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$, $q(x; \mu, \sigma^2)$ is the density of x.
Say we want to shift mean of X by $+\gamma$, so that $X \sim \mathcal{N}(\mu + \gamma, \sigma^2)$. Let $p(x; \mu + \gamma, \sigma^2)$ be the new density.

$$
\begin{aligned}
\frac{p(x; \mu + \gamma, \sigma^2)}{q(x; \mu, \sigma^2)} &= \frac{\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(\mathrm{x} - (\mu + \gamma))^2\right\}}{\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(\mathrm{x} - \mu)^2\right\}} \\
&= \frac{\exp\left\{-\frac{1}{2\sigma^2}((\mathrm{x} - \mu)^2 - 2\gamma(\mathrm{x} - \mu) + \gamma^2)\right\}}{\exp\left\{-\frac{1}{2\sigma^2}(\mathrm{x} - \mu)^2\right\}} \\
&= \exp\left\{\frac{\gamma(\mathrm{x} - \mu)}{\sigma^2} + \frac{\gamma^2}{2\sigma^2}\right\}
\end{aligned}
$$

Let $X \sim \mathcal{N}(\mu, \sigma^2)$, $q(x; \mu, \sigma^2)$ is the density of x.
Say we want to shift mean of X by $+\gamma$, so that $X \sim \mathcal{N}(\mu + \gamma, \sigma^2)$. Let $p(x; \mu + \gamma, \sigma^2)$ be the new density.

$$\begin{aligned}
\frac{p(x; \mu + \gamma, \sigma^2)}{q(x; \mu, \sigma^2)} &= \frac{\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x - (\mu + \gamma))^2\right\}}{\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\}} \\
&= \frac{\exp\left\{-\frac{1}{2\sigma^2}((x - \mu)^2 - 2\gamma(x - \mu) + \gamma^2)\right\}}{\exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\}} \\
&= \exp\left\{\frac{\gamma(x - \mu)}{\sigma^2} + \frac{\gamma^2}{2\sigma^2}\right\}
\end{aligned}$$

$B_t \sim \mathcal{N}(0, t)$, we want to shift mean by $-\int_0^t h(x_s)ds$.
Consider $\Delta B_t \sim \mathcal{N}(0, \Delta t)$, we want to shift mean by $-h(X_t)\Delta t$:

$$\begin{aligned}
\frac{p(\Delta b)}{q(\Delta b)} &= \exp\left\{-\frac{h(X_t)\Delta t \Delta b}{\Delta t} - \frac{1}{2}\frac{h(X_t)^2 \Delta t^2}{\Delta t}\right\} \\
&= \exp\left\{-h(X_t)\Delta b - \frac{1}{2}h(X_t)^2 \Delta t\right\}
\end{aligned}$$

Let

$$\tilde{D}_t := D_t^{-1} = \exp\left\{\int_0^t h(X_s)^* dB_s + \frac{1}{2}\int_0^t \|h(X_s)\|^2 ds\right\}$$
$$= \exp\left\{\int_0^t h(X_s)^* dY_s - \frac{1}{2}\int_0^t \|h(X_s)\|^2 ds\right\} = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$$

and define

$$\rho_t(\varphi) = \mathbb{E}_{\mathbb{P}}\left[\varphi(X_t)\tilde{D}_t \,\middle|\, \mathcal{Y}_t\right].$$

Can check that

$$\rho_t(1)\pi_t(\varphi) = \rho_t(\varphi).$$

Kallianpur-Striebel formula[9]:

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)}$$

[9] Ch. 3.4, Bain and Crisan, Stochastic Filtering Theory (2009)

Zakai equation

Let $\Gamma_t := \int_0^t h(X_s)^* dY_s - \frac{1}{2}\int_0^t \|h(X_s)\|^2 ds$.

By Itô's lemma,

$$d\tilde{D}_t = \tilde{D}_t d\Gamma_t + \frac{1}{2}\tilde{D}_t d\left\langle \Gamma \right\rangle_t = \tilde{D}_t h^*(X_t) dY_t,$$

and

$$d\varphi(X_t) = \mathcal{L}\varphi(X_t)dt + \nabla\varphi(X_t)dW_t,$$

where $\mathcal{L}$ is the generator of the Itô diffusion X.

Then,

$$\begin{aligned} d(\varphi(X_t)\tilde{D}_t) &= d\varphi(X_t)\tilde{D}_t + \varphi(X_t)d\tilde{D}_t + d\left\langle \varphi(X), \tilde{D} \right\rangle_t \\ d\rho_t(\varphi) &= \mathbb{E}_{\mathbb{P}}\left[d\varphi(X_t)\tilde{D}_t \middle| \mathcal{Y}_t \right] + \mathbb{E}_{\mathbb{P}}\left[\varphi(X_t)d\tilde{D}_t \middle| \mathcal{Y}_t \right] + \mathbb{E}_{\mathbb{P}}\left[d\left\langle \varphi(X_t), \tilde{D}_t \right\rangle \middle| \mathcal{Y}_t \right] \\ &= \rho_t(\mathcal{L}\varphi)dt + \rho_t(\varphi h^*)dY_t. \end{aligned}$$

Zakai equation

Let $u_t(x)$ be the density for the conditional expectation $\mathbb{E}_{\mathbb{P}}\left[\ \cdot|\mathcal{Y}_{0,t}\right]$ and $[\cdot,\cdot]$ be the inner product

$$[\varphi, u_t] = \int_{\mathbb{R}^m} \varphi(x) u_t(x) dx$$

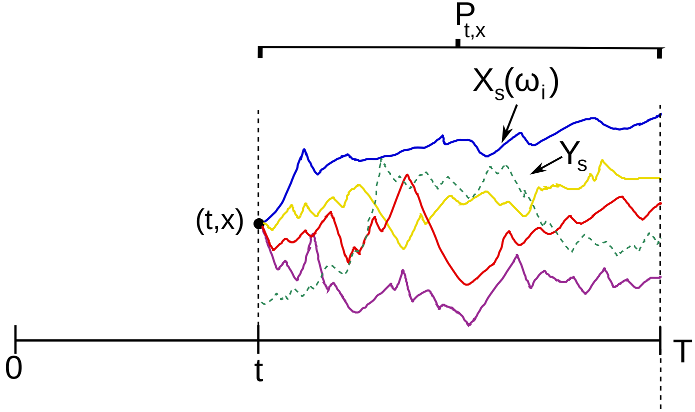
so $\rho_t(\varphi) = \frac{[\varphi, u_t]}{[1, u_t]}$.

Zakai equation:

$$\begin{aligned} d\rho_t(\varphi) &= \rho_t(\mathcal{L}\varphi)dt + \rho_t(\varphi h^*)dY_t \\ d[\varphi, u_t] &= [\mathcal{L}\varphi, u_t]dt + [\varphi h^*, u_t]dY_t \\ \Longrightarrow \quad du_t(x) &= \mathcal{L}^* u_t(x)dt + h(x)u_t(x)dY_t, \quad u_0(x) = q(x). \end{aligned}$$

Introduce a dynamic version of $\rho_T(\varphi) = \mathbb{E}_{\mathbb{P}}[\varphi(X_T^\varepsilon)\tilde{D}_T^\varepsilon | \mathcal{Y}_{0,T}^\varepsilon]$:

$$v_t^{T,\varphi}(x) = \mathbb{E}_{\mathbb{P}_{t,x}}[\varphi(X_T)\tilde{D}_{t,T} | \mathcal{Y}_{t,T}]$$



Introduce a dynamic version of $\rho_T(\varphi) = \mathbb{E}_{\mathbb{P}}[\varphi(X_T^\varepsilon)\tilde{D}_T^\varepsilon | \mathcal{Y}_{0,T}^\varepsilon]$:

$$v_t^{T,\varphi}(x) = \mathbb{E}_{\mathbb{P}_{t,x}}[\varphi(X_T)\tilde{D}_{t,T}|\mathcal{Y}_{t,T}]$$

By Markov property,

$$\begin{aligned}
\rho_t(v_t^{T,\varphi}(\cdot)) &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}_{t,X_t}}\left[\varphi(X_t)\tilde{D}_{t,T}\middle|\mathcal{Y}_{t,T}\right]\middle|\mathcal{Y}_{0,t}\right] \\
&= \int_{\mathbb{R}^m} v_t^{T,\varphi}(x)u_t(x)\,dx \\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(\zeta)u_{t,T}(\zeta;x)\,d\zeta\right)u_t(x)\,dx \\
&= \int_{\mathbb{R}^m}\varphi(x)u_T(x)\,dx = \rho_T(\varphi)
\end{aligned}$$

Dynamic version of $\rho_t(\varphi)$: $v_t^{T,\varphi}(x) = \mathbb{E}_{\mathbb{P}_{t,x}}[\varphi(X_T)\tilde{D}_{t,T}|\mathcal{Y}_{t,T}]$

Zakai equation:

$$\rho_t\left(v_{T-t}^{T,\varphi}\right) = \rho_0\left(v_T^{T,\varphi}\right) + \int_0^t \rho_s\left(\mathcal{L}v_{T-s}^{T,\varphi}\right)ds + \int_0^t \rho_s\left(v_{T-s}^{T,\varphi}h^*\right)dY_s$$

The dual equation for the dynamic version of ρ_t can be obtained as

$$v_{T-t}^{T,\varphi}(x) = \varphi(x) + \int_{T-t}^T \mathcal{L}v_s^{T,\varphi}(x)ds + \int_{T-t}^T v_s^{T,\varphi}(x)h^*(x)d\overleftarrow{Y}_s$$

where Y is a $\mathbb{P}$-Brownian motion independent of the signal noise W and $\int \cdot\, d\overleftarrow{Y}$ is a backward stochastic integral.

For rigorous treatment of this backward SPDE, see *Pardoux, Stochastic Partial Differential Equations and Filtering of Diffusion Processes (1979)*.

Backward stochastic PDE

$$\psi(\omega,t,x) = \psi(\omega,T,x) + \int_t^T \{\mathcal{L}\psi(\omega,s,x)ds + f(\omega,s,x)\}\, ds$$
$$+ \int_t^T \{g(\omega,s,x) + G(\omega,s,x)\psi(\omega,s,x))\}\, d\overleftarrow{B}_s,$$
$$\psi(\omega,T,x) = \Psi(T,x)$$

The solution is adapted to $\mathcal{F}^B_{t,s}$, the completion of the filtration

$$\mathcal{F}^{0,B}_{t,s} \stackrel{\text{def}}{=} \bigcap_{r<t} \sigma\left(B_u - B_r : r \le u \le s\right)$$

Recall, for $s \in (T-t, T]$, $H'_s := H_{T-s}$ and $B'_s := B_T - B_{T-s}$,

$$\int_{T-t}^{T} H_s d\overleftarrow{B}_s = \int_0^t H'_s dB'_s.$$

Also,

$$\int_0^t H'_s\, ds = \int_0^t H_{T-s}\, ds = -\int_T^{T-t} H_\tau\, d\tau = \int_{T-t}^{T} H_\tau\, d\tau.$$

Let $\psi'(t,x) = \psi(T-t,x)$.
Backward spde:

$$\begin{aligned}\psi(\omega, T-t, x) = \Psi(T,x) &+ \int_{T-t}^{T} \{\mathcal{L}\psi(\omega,s,x)ds + f(\omega,s,x)\}\, ds \\ &+ \int_{T-t}^{T} \{g(\omega,s,x) + G(\omega,s,x)\psi(\omega,s,x))\}\, d\overleftarrow{B}_s\end{aligned}$$

Recall, for $s \in (T-t, T]$, $H'_s := H_{T-s}$ and $B'_s := B_T - B_{T-s}$,

$$\int_{T-t}^{T} H_s d\overleftarrow{B}_s = \int_0^t H'_s dB'_s.$$

Also,

$$\int_0^t H'_s\, ds = \int_0^t H_{T-s}\, ds = -\int_T^{T-t} H_\tau\, d\tau = \int_{T-t}^{T} H_\tau\, d\tau.$$

Let $\psi'(t,x) = \psi(T-t,x)$.
Forward version of the backward spde:

$$\psi'(\omega,t,x) = \Psi(T,x) + \int_0^t \{\mathcal{L}\psi'(\omega,s,x) + f(\omega,s,x)\}\, ds + \int_0^t \{g(\omega,s,x) + G(\omega,s,x)\psi'(\omega,s,x)\}\, dB'_s$$

For the backward spde, we can use existence and uniqueness and other results for forward spdes.

SDE representation of BSPDE solution

Let $\mathcal{L}$ be the generator of the following diffusion process:

$$X_s^{t,x} = x + \int_t^s b(X_s^{t,x})ds + \int_t^s \sigma(X_s^{t,x})dW_s \qquad \text{for } s \geq t,$$
$$X_s^{t,x} = x \qquad \text{for } s \leq t.$$

For $s \in [t, T]$, define a stochastic version of $\psi(\omega, s, x)$: $\psi(s, X_s^{t,x})$.

Partition $[t, T]$ into N intervals $(t_0, t_1], (t_1, t_2], \ldots, (t_{N-1}, t_N]$,
$t_k := T - (N - k) \left\lfloor \frac{T-t}{N} \right\rfloor$.

$$\psi(t, X_t^{t,x}) = \psi(T, X_T^{t,x}) + \lim_{N \nearrow \infty} \sum_{i=0}^{N-1} (\psi(t_i, X_{t_i}^{t,x}) - \psi(t_{i+1}, X_{t_{i+1}}^{t,x}))$$

and

$$\begin{aligned}
&\psi(t_i, X_{t_i}^{t,x}) - \psi(t_{i+1}, X_{t_{i+1}}^{t,x}) \\
&\quad = (\psi(t_i, X_{t_i}^{t,x}) - \psi(t_i, X_{t_{i+1}}^{t,x})) + (\psi(t_i, X_{t_{i+1}}^{t,x}) - \psi(t_{i+1}, X_{t_{i+1}}^{t,x})) \\
&\quad = -\left(\int_{t_i}^{t_{i+1}} \mathcal{L}\psi(t_i, X_s^{t,x}) ds + \int_{t_i}^{t_{i+1}} \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \psi(t_i, X_s^{t,x}) \sigma_{ij}(X_s^{t,x}) dW_s^j \right) \\
&\qquad + \int_{t_i}^{t_{i+1}} (\mathcal{L}\psi(s, X_{t_{i+1}}^{t,x}) + f(s, X_{t_{i+1}}^{t,x})) ds \\
&\qquad + \int_{t_i}^{t_{i+1}} (g(s, X_{t_{i+1}}^{t,x}) + G(X_{t_{i+1}}^{t,x}) \psi(s, X_{t_{i+1}}^{t,x})) d\overleftarrow{B}_s.
\end{aligned}$$

$$\psi(t, X_t^{t,x}) = \psi(T, X_T^{t,x}) + \lim_{N \nearrow \infty} \sum_{i=0}^{N-1} (\psi(t_i, X_{t_i}^{t,x}) - \psi(t_{i+1}, X_{t_{i+1}}^{t,x}))$$

and

$$\begin{aligned}
\psi(t_i, & X_{t_i}^{t,x}) - \psi(t_{i+1}, X_{t_{i+1}}^{t,x}) \\
&= (\psi(t_i, X_{t_i}^{t,x}) - \psi(t_i, X_{t_{i+1}}^{t,x})) + (\psi(t_i, X_{t_{i+1}}^{t,x}) - \psi(t_{i+1}, X_{t_{i+1}}^{t,x})) \\
&= -\left(\int_{t_i}^{t_{i+1}} \mathcal{L}\psi(t_i, X_s^{t,x}) ds + \int_{t_i}^{t_{i+1}} \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \psi(t_i, X_s^{t,x}) \sigma_{ij}(X_s^{t,x}) dW_s^j \right) \\
&\quad + \int_{t_i}^{t_{i+1}} (\mathcal{L}\psi(s, X_{t_{i+1}}^{t,x}) + f(s, X_{t_{i+1}}^{t,x})) ds \\
&\quad + \int_{t_i}^{t_{i+1}} (g(s, X_{t_{i+1}}^{t,x}) + G(X_{t_{i+1}}^{t,x}) \psi(s, X_{t_{i+1}}^{t,x})) d\overleftarrow{B}_s.
\end{aligned}$$

As $N \nearrow \infty$, for $s \in [t, T)$,

$$\begin{aligned}
d\psi(s, X_s^{t,x}) = f(s, X_s^{t,x}) ds + \left(g(s, X_s^{t,x}) + G(X_s^{t,x}) \psi(s, X_s^{t,x}) \right) d\overleftarrow{B}_s \\
+ \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \psi(t, X_s^{t,x}) \sigma_{ij}(X_s^{t,x}) dW_s^j
\end{aligned}$$

SDE representation of BSPDE solution

Backward doubly-stochastic differential equation:

$$-dY_s^{t,x} = f(s, X_s^{t,x})ds + (g(s, X_s^{t,x}) + G(s, X_s^{t,x})Y_s^{t,x})d\overleftarrow{B}_s - Z_s^{t,x}dW_s,$$
$$Y_T^{t,x} = \Psi(T, X_T^{t,x}).$$

where $Y_s^{t,x} = \psi(s, X_s^{t,x})$, $Z_s^{t,x} = \sum_{j=1}^k \sum_{i=1}^m \frac{\partial}{\partial x_i}\psi(t, X_s^{t,x})\sigma_{ij}(X_s^{t,x})$ and

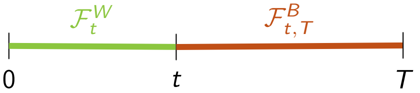
$$X_s^{t,x} = x + \int_t^s b(X_s^{t,x})ds + \int_t^s \sigma(X_s^{t,x})dW_s \qquad \text{for } s \geq t,$$
$$X_s^{t,x} = x \qquad \text{for } s \leq t.$$

SDE representation of BSPDE solution

Backward doubly-stochastic differential equation:

$$\begin{aligned} -dY_s^{t,x} &= f(s, X_s^{t,x})ds + (g(s, X_s^{t,x}) + G(s, X_s^{t,x})Y_s^{t,x})d\overleftarrow{B}_s - Z_s^{t,x}dW_s, \\ Y_T^{t,x} &= \Psi(T, X_T^{t,x}). \end{aligned}$$

$Y_s^{t,x}$ has to be measurable w.r.t. $\mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$



*Not a filtration, doesn't increase with time